Science Journal of Applied Mathematics and Statistics
2023; 11(3): 44-47
http://www.sciencepublishinggroup.com/j/sjams
doi: 10.11648/j.sjams.20231103.11
ISSN: 2376-9491 (Print); ISSN: 2376-9513 (Online)

Improvement of the Raabe-Duhamel Convergence Criterion Generalized

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To cite this article:

El Hadji Abdoulaye Thiam, Papa Moussa Niang. Improvement of the Raabe-Duhamel Convergence Criterion Generalized. *Science Journal of Applied Mathematics and Statistics*. Vol. 11, No. 3, 2023, pp. 44-47. doi: 10.11648/j.sjams.20231103.11

Received: July 18, 2023; **Accepted:** August 23, 2023; **Published:** November 16, 2023

Abstract: In Mathematics, convergence tests are methods of testing for the convergence, conditional convergence, absolute convergence, interval of convergence or divergence of an infinite series. There are many criterion for testing the convergence of an infinite series: Cauchy, D'Alembert, Riemann, Bertrand and so one. One of the most important is the Raabe-Duhamel convergence criterion which asserts that: Given an infinite series $\sum_n u_n$ with positive terms u_n and assuming that the following expansion holds $\frac{u_{n+1}}{u_n} = 1 - \frac{\lambda}{n} + o(1/n)$, as $n \to \infty$. Then the series $\sum_n u_n$ converges if $\lambda > 1$ and diverges if $\lambda < 1$. However no conclusion can be made if $\lambda = 1$. Indeed the infinite series $\sum_n \frac{1}{n}$ and $\sum_n \frac{1}{n(\log(n))^2}$ satisfy both the expansion with $\lambda = 1$. The first one converges and the second one diverges. The aim of the present paper deals with the convergence of a generalized Riemann-Bertrand infinite series. This will allows us to improve the expansion so that something can be said if $l = 1$: this corresponds to the improvement of the Raabe-Duhamel convergence criterion. This improvement is based on the convergence of a new type of infinite series. These type of series are generalization of the Riemann and Bertrand infinite series.

Keywords: Infinite Series, Raabe-Duhamel Convergence Criterion, Riemann and Bertrand Infinite Series

1. Introduction

Roughly speaking, a series is a description of the operation of adding infinitely many quantities to a given starting quantity. The study of series is a major part of calculus and its generalization, mathematical analysis. Series are used in most areas of mathematics, even for studying finite structures through generating functions. In addition, infinite series are also widely used in other quantitative disciplines such as physics, computer science, statistics and finance.

One of the most known infinite series in the litterature is the Riemann one:

$$\sum_{n\geq 1} \frac{1}{n^{\gamma_0}} \tag{1}$$

where γ_0 is a real number. This series is convergent if and only if $\gamma_0 > 1$. Another important one is the Bertrand series:

$$\sum_{n\geq 2} \frac{1}{n^{\gamma_0}(\log(n))^{\gamma_1}}, \tag{2}$$

where γ_0 and γ_1 are real numbers. This series converges if and only if $\gamma_0 > 1$ or $(\gamma_0 = 1$ and $\gamma_1 > 1)$, see for instance [[2] Proposition 1.8 and Proposition 1.10].

In this paper, we are interested in a new extension of this type of series. More precisely, we condider the following infinite series

$$\sum_{n\geq 2} \frac{1}{n^{\gamma_0}(\log(n))^{\gamma_1}(\log(\log(n)))^{\gamma_2}}, \tag{3}$$

where γ_0, γ_1 and γ_2 are real numbers. Then we have the following result.

Theorem 1.1. Consider the infinite series

$$\sum_{n\geq 3} \frac{1}{n^{\gamma_0}(\log(n))^{\gamma_1}(\log(\log(n)))^{\gamma_2}}.$$

Then it is convergent if and only if one of these three conditions is satisfied:

1. Case 1: $\gamma_0 > 1$.

2. Case 2: $\gamma_0 = 1$ et $\gamma_1 > 1$.
3. Case 3: $\gamma_0 = \gamma_1 = 1$ et $\gamma_2 > 1$.

Moreover we extend the result as follows.

Theorem 1.2. Let $k \in \mathbb{N}$ fixed, $n \in \mathbb{N}$ large enough and consider the following infinite series

$$\sum_n \frac{1}{n^{\gamma_0}\prod_{i=1}^{k}(f^{(i)}(n))^{\gamma_i}},$$

where

$$f^{(1)}(\cdot) = \log(\cdot) \text{ and } f^{(i)}(\cdot) = \underbrace{fo\cdots of(\cdot)}_{i-times}.$$

Then it converges if and only if one of these conditions is satisfied

1. $\gamma_0 > 1$.
2. $\gamma_0 = \cdots = \gamma_i = 1$ and $\gamma_{i+1} > 1$ for $i = 1, \cdots, k-1$.

Another important part of infinite series is the convergence criterions such as Bertrand, Gauss, Cauchy, Raabe-Duhamel etc. and the convergence rules. Most of the time, they are consequences of some comparison theorems and convergences of some special infinites series, Riemann, Bertrand and so one. In this dynamique, as a consequence of Theorem 1.1 and Theorem 1.2, we improve the famous Raabe-Duhamel convergence criterion. For that we consider a positive real infinite series $\sum_n u_n$, satisfying

$$\frac{u_{n+1}}{u_n} = 1 - \frac{\gamma_0}{n} - \sum_{i=1}^{k}\frac{\gamma_i}{n\prod_{i=1}^{k} f^{(i)}(n)} + o\left(\frac{1}{n\prod_{i=1}^{k} f^{(i)}(n)}\right),$$

as $n \to \infty$, where the $\{\gamma_i\}_{0\le i\le k}$ are real numbers. Depending on the values of the coefficients $\{\gamma_i\}_{0\le i\le k}$, we can deduce the convergence of the infinite series. This is an improvement of the famous Raabe-Duhamel convergence criterion, see for instance Section 3. For more details related to recent convergence criterions (Bertrand, Raabe-Duhamel, Kummel etc.), we refer to [7]. The literature about convergence of infinite series and related subject is very huge. For more detaiuls related to that we refer to [1-6, 8, 10-12, 14, 15] and references therein.

The paper is organized as follows. In Section 2, we give a complete proof of Theorem 1.1 and Theorem 1.2 and Section 3 is devoted to applications of to the Raabe-Duhamed convergence criterion.

2. Proof of the Main Results

Proof of *Theorem 1.1*.

We let $n \ge 3$ and we consider the sequence

$$u_n = \frac{1}{n^{\gamma_0}(\log(n))^{\gamma_1}(\log(\log(n)))^{\gamma_2}},$$

where $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}$.

Case 1:

Setting

$$v_n = \frac{1}{n^{\frac{1+\gamma_0}{2}}},$$

we have

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \begin{cases} 0 & \text{if } \gamma_0 > 1 \\ +\infty & \text{if } \gamma_0 < 1. \end{cases}$$

Consequently, the series $\sum_n u_n$ converges if $\gamma_0 > 1$ and diverges if $\gamma_0 < 1$.

Case 2:

Now we assume that $\gamma_0 = 1$ and consider the Bertrand series $\sum w_n$ given by

$$w_n = \frac{1}{n(\log(n))^{\frac{1+\gamma_1}{2}}}.$$

Then we have

$$\lim_{n\to\infty}\frac{u_n}{w_n} = \begin{cases} 0 & \text{if } \gamma_1 > 1 \\ +\infty & \text{if } \gamma_1 < 1. \end{cases}$$

Consequently the series $\sum_n u_n$ converges if $\gamma_1 > 1$ and diverges if $\gamma_1 < 1$.

Case 3.1: In the last case, we first assume that $\gamma_0 = \gamma_1 = 1$ and $\gamma_2 < 0$. Then

$$\lim_{n\to\infty}\frac{u_n}{\frac{1}{n(\log(n))}} = \lim_{n\to+\infty}\frac{1}{(\log(\log(n)))^{\gamma_2}} = +\infty.$$

Therefore the series $\sum_n u_n$ diverges.

Case 3.2: We now assume that $\gamma_0 = \gamma_1 = 1$ and $\gamma_2 \ge 0$. The map

$$t \in [3, +\infty) \longmapsto f(t) = \frac{1}{t\log(t)(\log(\log(t)))^{\gamma_2}}$$

is decreasing. Therefore the generalized integral

$$\int_3^\infty f(t)dt$$

and the infinite series $\sum_n u_n$ have the same nature.

By the change of variable formula $u = \log(t)$, we get

$$\int_3^\infty f(t)dt = \int_{\log(3)}^\infty \frac{du}{u(\log(u))^{\gamma_2}}.$$

The last integral is a Bertrand one of cofficients $(1, \gamma_2)$. Thanks to [[3], Chapter 9], it converges if and only if $\gamma_2 > 1$. This then ends the proof.

Proof of *Theorem 1.2*.

The proof of Theorem 1.2 uses the same argument as before. By induction, we successively compare the corresponding infinite series to the inifnite series of Riemann (1), Betrand (2), the one in Theorem 1.1 (3) etc. For the finale case

$$\gamma_0 = \gamma_1 = \cdots = \gamma_{k-1} = 1,$$

we proceed as in *Case 3.1* and *Case 3.2*.

In the last case, we first assume that $\gamma_0 = \cdots = \gamma_{k-1} = 1$ and $\gamma_k < 0$. Then

$$\lim_{n\to\infty} \frac{u_n}{\dfrac{1}{n\displaystyle\prod_{i=1}^{k-1}(f^{(i)}(n))}} = \lim_{n\to+\infty} \frac{1}{(f^{(k)}(n))^{\gamma_k}} = +\infty.$$

Therefore the series $\sum_n u_n$ diverges because in the previous step, the infinite series

$$\sum_n \frac{1}{n\displaystyle\prod_{i=1}^{k-1}(f^{(i)}(n))}$$

diverges.

We now assume that $\gamma_0 = \cdots = \gamma_{k-1} = 1$ and $\gamma_k \geq 0$. Then, for c large enough, the map

$$t \in [c, +\infty) \longmapsto f(t) = \frac{1}{t\displaystyle\prod_{i=1}^{k-1}(f^{(i)}(t)}$$

is decreasing.

Therefore

$$\int_c^\infty f(t)dt$$

and the infinite series $\sum_n u_n$ have the same nature.

By the change of variable formula $u = f^{(k-1)}(t)$, we get

$$\int_c^\infty f(t)dt = \int_{f^{(k-1)}(c)}^\infty \frac{du}{u(\log(u))^{\gamma_k}}.$$

The last integral is a Bertrand one of cofficients $(1, \gamma_k)$. Thanks to [[3], Chapiter 9], it converges if and only if $\gamma_k > 1$. This then ends the proof.

3. Some Applications: Improvement of Raabe-Duhamel Convergence Criterion

The following result is the key tool for the proof of the following results. Then we have:

Lemma 3.1. Let $(u_n)_n$ and $(v_n)_n$ two positive real sequences satisfying

$$\exists N \in \mathbb{N}, \forall n \in \mathbb{N} \quad \left(n \geq N \Rightarrow \frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}\right).$$

Then

1. If the series $\sum_n v_n$ converges then the series $\sum_n u_n$ converges.
2. If the series $\sum_n u_n$ diverges then the series $\sum_n v_n$ diverges.

For the proof, we refer to [[2], exercice 1.6].

Corollary 3.1. Let $\sum u_n$ be a series of positive real numbers such that

$$\frac{u_{n+1}}{u_n} = 1 - \frac{\gamma_0}{n} + o\left(\frac{1}{n}\right), \qquad \text{as } n \to \infty, \tag{4}$$

where γ_0 is a real number.

Then $\sum u_n$ converges if $\gamma_0 > 1$ and diverges if $\gamma_0 < 1$.

Remark 3.1. .

1. If $u_n = \frac{1}{n}$ or $u_n = \frac{1}{n(\log(n))^2}$, then we have

   $$\frac{u_{n+1}}{u_n} = 1 - \frac{1}{n} + o\left(\frac{1}{n}\right), \qquad \text{as } n \to \infty.$$

   Then, no conclusion can be definite in the case $\gamma_0 = 1$.
2. However an improving of this expansion, will be helpfull concerning the case $\gamma_0 = 1$. This is the aim of the following results. We refer the reader to [7] for more results related to this improvement.

We begin by this particular case which is a direct consequence of Theorem 1.1. Then we have

Corollary 3.2. let $\sum u_n$ be a series of positive real numbers such that

$$\frac{u_{n+1}}{u_n} = 1 - \frac{1}{n} - \frac{\gamma_1}{n\log(n)} + o\left(\frac{1}{n\log(n)}\right), \qquad \text{as } n \to \infty$$

where $\gamma_1 \in \mathbb{R}$. Then the series $\sum_n u_n$ converges if $\gamma_1 > 1$ and diverges if $\gamma_1 < 1$.

Proof We set

$$v_n = \frac{1}{n(\log(n))^\lambda},$$

where $\lambda = \frac{1+\gamma_1}{2}$.

Then we easily see that

$$\frac{v_{n+1}}{v_n} = 1 - \frac{1}{n} - \frac{\lambda}{n\log(n)} + o\left(\frac{1}{n\log(n)}\right), \qquad \text{as } n \to \infty. \tag{5}$$

Then the result follows immediately from Lemma 3.1 and (5). This then ends the proof.

We finish by the following generalization.

Corollary 3.3. Let $\sum u_n$ be a series of positive real numbers such that

$$\frac{u_{n+1}}{u_n} = 1 - \frac{\gamma_0}{n} - \sum_{i=1}^{k} \frac{\gamma_i}{n\prod_{i=1}^{k} f^{(i)}(n)} + o\left(\frac{1}{n\prod_{i=1}^{k} f^{(i)}(n)}\right),$$

as $n \to \infty$,
where the $\{\gamma_i\}_{0\leq i\leq k}$ are real numbers. Then the series $\sum_n u_n$

(i) converges if ($\gamma_0 > 1$) or ($\gamma_0 = \cdots = \gamma_i = 1$ and $\gamma_{i+1} > 1$) for $i = 1, \cdots k-1$

and

(ii) diverges if ($\gamma_0 < 1$) or ($\gamma_0 = \cdots = \gamma_i = 1$ and $\gamma_{i+1} < 1$) for $i = 1, \cdots, k-1$.

Proof For n large enough, we set

$$v_n = \frac{1}{n^{\lambda_0}\prod_{i=1}^{k}(f^{(i)}(n))^{\lambda_i}},$$

where

$$\lambda_i = \frac{1+\gamma_i}{2}, \qquad \text{for all } i = 0, \cdots, k.$$

Then we have

$$\frac{v_{n+1}}{v_n} = 1 - \frac{\lambda_0}{n} - \sum_{i=1}^{k} \frac{\lambda_i}{n\prod_{i=1}^{k} f^{(i)}(n)} + o\left(\frac{1}{n\prod_{i=1}^{k} f^{(i)}(n)}\right),$$

as $n \to \infty$.

Then the result follows immediately from Theorem 1.1 and Lemma 3.1.

4. Conclusion

In this paper, we have studied a generalization of the infinite series of Bertrand which is also a generalization of the infinite series of Riemann. This then allows us to extend the expansion in $\frac{u_{n+1}}{u_n} = 1-\frac{\lambda}{n}+o(1/n)$ and then improve the famous Raabe-Duhamel convergence criterion.

Acknowledgments

The authors thank the anonymous referees for a careful reading and some helpful comments, which greatly improve the manuscript.

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