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Approximation of Functions Using Fourier Series and Its Application to the Solution of Partial Differential Equations

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Abstract: Fourier series are a powerful tool in applied mathematics; indeed, their importance is twofold since Fourier series are used to represent both periodic real functions as well as solutions admitted by linear partial differential equations with assigned initial and boundary conditions. The idea inspiring the introduction of Fourier series is to approximate a regular periodic function, of period T, via a linear superposition of trigonometric functions of the same period T; thus, Fourier polynomials are constructed. They play, in the case of regular periodic real functions, a role analogue to that one of Taylor polynomials when smooth real functions are considered. In this thesis we will study function approximation by FS method. We will make an attempt to approximate square wave function, line function by FS, and line function by Fourier exponential and trigonometric polynomial. DFT will also be used to approximate function values from data set. We compare the accuracy and the error of Fourier approximation with the actual function and we find that the approximate function is very close to the actual function. We also study the solution of 1D heat equation and Laplace equation by Fourier series method. We compare the solution of heat equation obtained by Fourier series with BTCS. We also compare the solution of Laplace equation obtained by Fourier series with Jacobi iterative method. MATLAB codes for each scheme are presented in appendix and results of running the codes give the numerical solution and graphical solution.

Keywords: Fourier Series, Sine Wave, Discrete Fourier Transform, Heat Equation and Laplace Equation

1. Introduction

1.1. Background of the Study

The theory of Function approximation is generally referred to as the representation of a function as close as possible to the actual function. There are many methods available to approximate functions, such as Least Square approximation, Chebyshev Polynomial approximation, Taylor series, Fourier series etc. One common way of approximating functions is to use Taylor series expansions. This relies on the computation of the Taylor polynomials of the function up to a certain order, and approximating the given function through these Taylor polynomials [3]. While this is a relatively simple procedure in case of smooth functions, it cannot work for non differentiable continuous functions. Also the convergence of this approximation is not uniformly distributed on a given interval and towards the end of the interval the approximation error is higher. In order to avoid these problems, one can use families of orthogonal polynomials like Fourier series.

This project will deals with approximation of functions by Fourier series, which uses the sum of trigonometric periodic functions to approximate function to almost exact precision. This tactic will result in minimal error when comparing it to the original function and the process is highly effective for continuous functions, but involves a larger error when handling discontinuous functions [7]. They are usually the best way to represent periodic function, something that cannot be done with a polynomial or a Taylor series and can even approximate functions with discontinuous and discontinuous derivatives. Joseph Fourier was a French mathematician who first proposed FS and their application to problem of heat transfer and vibration in the early 1800s [17]. He state that "any function can be represented by infinite sum of sine and cosine terms" Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots \quad (1)$$

Where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \quad (2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx,\ n = 0, 1, 2, \ldots \quad (3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx,\ n = 1, 2, 3, \ldots \quad (4)$$

Fourier series are an important topic in mathematics due to their high application in Physical sciences, Engineering and other applied science. They are used to approximate complex functions in many different parts of science and mathematics. The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to original problem or an approximation to it whatever accuracy is desired [5]. They are helpful in their ability to imitate many different types of waves: x-ray, heat, light and sound. Fourier analysis is an essential component of much of modern applied and pure mathematics. It forms an exceptionally powerful tool for solving a broad range of Ordinary and Partial differential equations. Fourier analysis lies at the heart of signal processing, including audio, speech, images, videos, radio transmissions, and so on [11].

In recently study, the finite Fourier coefficients provide a good approximation to the Fourier coefficients of a piecewise continuous function. For a continuous periodic function, the size of the error is estimated in terms of the modulus of continuity of the function. The estimates improve commensurately as the functions become smoother (Wiley, 2005). And also the partial sums of the finite Fourier transform provide essentially as good an approximation to the function and its derivatives as the partial sums of the ordinary Fourier series [13, 15]. Along the way we establish analogues of the Riemann Lebesgue lemma and the localization principle [6]. The focus in the work of [4] is on how to use the exact knowledge of a finite number of Fourier coefficients to obtain a very accurate approximation to a piecewise analytic function, even when it has jump discontinuities. In [1] methods are described for approximating Fourier coefficients outside of the range of minimum and maximum of constant These techniques involve using higher order interpellants to approximate the function from the sampled data [9] and consider the problem of estimating the error by using the discrete Fourier transform to compute the Fourier transform of a square integrable function. Using Cauchy-Schwarz inequality, the authors derive relative bounds for the errors in the Fourier coefficients that depend on the sampling models and the frequency but are independent of the function [8].

1.2. Statement of the Problem

In this thesis an approximation of a function $f(x)$, can be obtained as the limit of a partial sum of a Fourier series, if we know in advance that the series converges to $f(x)$. The situation is different when we have not succeeded in proving that the series converges or when the series turns out to be divergent, for then either we do not know whether or not the partial sums have a limit or else we actually know that the limit does not exist. Thus we have to find an operation which allows us to determine a function from knowledge of its Fourier series, regardless of whether or not the series converges. This is the problem that we concern in this project.

The coefficients of trigonometric Fourier series will be calculated or determined based on the property of orthogonality for sines and cosine and compute the inner product of the basis. [10]. As the functions become more and more complicated the numerical integration is used to evaluate the integral. This is done using MATLAB program. The approximation function is determined by the coefficients of the trigonometric polynomial. The least squares condition is used to select these coefficients. That is, the coefficients are determined by minimizing the integral of the square of the difference between the approximation. This project also deals with the application of Fourier series to Partial differential equations that arise in mathematical physics, Such as heat equation, wave equation Laplace equation that is covered in more advanced courses [22]. The first part of this project is an introduction of the general nature of Fourier series with properties, definitions, and theorems. In the second part we approximate by Fourier series periodic and non-periodic functions using MATLAB and the approximate function is compared with the actual function, and discussion of the results is given. Lastly we discuss the approximation of solution by FS in PDEs using MATLAB.

1.3. Objectives of the Study

1.3.1. General Objective

The general objective of this thesis is to approximate functions by Fourier series using MATLAB such that the result is as close as possible to the actual function.

1.3.2. Specific Objectives

This project will go through with the following specific objectives.

1. To approximate periodic and non-periodic functions by trigonometric polynomial.
2. To approximate solution of PDEs using FS.
3. To compare the numerical solution of implicit BTCs method with Fourier series method for one dimension Heat equation.
4. To compare the solution of Laplace equation obtained by FS with Jacobi iterative method.
5. To discuss the errors in approximation.

2. Literature Review

2.1. Fourier Series

Mathematicians of the eighteenth century, including

Daniel Bernoulli and Leonard Euler, expressed the problem of the vibratory motion of a stretched string through partial differential equations that had no solutions in terms of elementary functions Christopher, J. Z, (2004). Their resolution of this difficulty was to introduce infinite series of sine and cosine functions that satisfied the equations. In the early nineteenth century, Joseph Fourier, while studying the problem of heat flow, developed a cohesive theory of such series [21]. Consequently, they were named after him. Fourier series and Fourier integrals are investigated [16]. Fourier series are infinite series that represent periodic functions in terms of cosine and sine function. Function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x, except possibly at some points, and if there is some positive number p, called a period of $f(x)$, such that

$$f(x+np)=f(x), n=1,2,3,\ldots \tag{5}$$

The mathematical expression of Fourier series for periodic function f(x) is:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)=f(x\pm 2L)=f(x\pm 4L)=\cdots \tag{6}$$

where a_0 , a_n and b_n are Fourier coefficients, to be determined by the following integrals:

$$a_n=\frac{1}{L}\int_0^{2L}f(x)\cos\frac{n\pi x}{L}dx, n=0,1,\ldots \tag{7}$$

$$b_n=\frac{1}{L}\int_0^{2L}f(x)\sin\frac{n\pi x}{L}dx, n=1,2, \tag{8}$$

Fourier series for a periodic function with period $(-\pi,\pi)$:

By using Equations (5) and (6) and (7), we will have Fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\propto}(a_n\cos nx+b_n\sin nx) \tag{9}$$

With

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx \tag{10}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\ n=0,1,2,\ldots. \tag{11}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx\ n=1,2,3,\ldots. \tag{12}$$

Fourier series is used for signal analysis and system design in modern telecommunications, and image processing systems. It is a family of mathematical techniques, all based on decomposing sound signals into sinusoids because sound signal is a segment of different frequencies [14]. Fourier series provides an alternative way of representing signal amplitude as function of frequency, which represent signal by how much information is contained at different frequencies (Attia, 1999; Boggess and Narcowich, 2009; Mandal and Asif, 2007). A Fourier series takes a signal and splits it into a sum of sinusoids of different frequencies (Jeffrey, 2002; karris, 2004). This means that the Fourier transform provides a method of transforming infinite duration signals from the time domain into continuous frequency domain [19]. If $f(t)$ is a periodic function of period of 2π then the function $f(t)$ is given in the expression of the form:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(n\omega t)+b_n\sin(n\omega t)) \tag{13}$$

Equation (13) is a trigonometric form of the Fourier series. The constant in the expression a_n where n= $0,1,2,3,\ldots$ b_n where n $=1,2,3,\ldots$ is determined by,

$$\left.\begin{aligned}\frac{a_0}{2}&=\frac{1}{2\pi}\int_0^{2\pi}f(t)dt\\ a_n&=\frac{1}{2\pi}\int_0^{2\pi}f(t)\cos(n\omega t)\,dt\\ b_n&=\frac{1}{2\pi}\int_0^{2\pi}f(t)b_n\sin(n\omega t)\,dt\end{aligned}\right\} \tag{14}$$

There are other forms of the Fourier series given in the form of Euler formula (Jeffrey, 2002; karris, 2004) which is written as

$$f(t)=\sum_{-\infty}^{\infty}c_n e^{jn\omega t} \tag{15}$$

Where

$$c_n=\frac{1}{2\pi}\int_0^{2\pi}f(t)\,e^{-jn\omega t}\,dt \tag{16}$$

Equation (15) is the complex Fourier series and (16) is coefficient of complex Fourier series.

2.2. Discrete Fourier Transform

Discrete Fourier transform is extremely important in the area of frequency analysis because it transforms a discrete signal in time domain to its discrete frequency domain representation. It decomposes sampled signals in terms of the sinusoidal complex exponential components (Attia, 1999; Elali, 2005; Orfanidis, 2010) such discrete time to discrete frequency transformation is enssential. In many signal processing applicable discrete Fourier transformation (DFT) has central role in which time series x[n] is the sum of the average component of the sampled signal and a series of sinusoidal with different amplitudes and frequencies (Mussoko, 2005). To show the frequency content of a signal we use the DFT (Madal and Asif, 2007; Orfanidis, 2010; Rocchesso, 2003)

$$X[\omega]=\sum_{-\infty}^{\infty}x[n]e^{-j\omega n} \tag{17}$$

Since the x[n] is finite length signal, this implicies that the DFT can be written as

$$X[\omega]=\sum_{n=0}^{N-1}x[n]e^{-j\omega n} \tag{18}$$

For a finite number of frequency points the above equation can be simplified to

$$X[\omega]= \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} \tag{19}$$

Where

$\omega_k = 2\pi k/N$ is the frequency sample at K values assuming that there are N samples with $k = 0, 1, 2, \ldots, N-1$. Hence $X[\omega]$ can be expressed as

$$X[\omega]= \sum_{n=0}^{N-1} x[n]e^{-\frac{j2\pi kn}{N}} \tag{20}$$

Then the N –point DFT X[k] is written as

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-\frac{J2\pi kn}{N}} \tag{21}$$

Here X[k] sampled version of X $[\omega]$ at $\omega = 2\pi k/N$. this can be written as

$$X[k] = \sum_{n=0}^{N-1} x[n]{W_N}^{kn} \tag{22}$$

$$\text{Where } W_N = e^{-\frac{2\pi kn}{N}} \tag{23}$$

The fact W, also known as the twiddle factor, is a function of N frequency terms with Kn which can take an integer values up to $(N-1)^2$ (Hsu, 1995; Wong, 2006). Each point of the DFT in equation (23) above can be calculated using N complex multiplications and $N-1$ complex addition. Therefore N^2 complex multiplication and $N(N-1)$ complex addition are to compute N number of DFT (Orfanidis, 2010; Rocchesso, 2003). We can express the N data points as

$$X_n = W_n x_n \tag{24}$$

Where W_n is the $N \times N$ matrix of linear transformation, is x_n is the N-point of the signal x[n] and X_n N point vector of frequency samples defined by:

$$w_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_N & {w_N}^2 & \cdots & {w_N}^{N-1} \\ 1 & {w_N}^2 & {w_N}^4 & \cdots & {w_N}^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & {w_N}^{N-1} & {w_N}^{2(N-1)} & \cdots & {w_N}^{(N-1)^2} \end{bmatrix} \tag{25}$$

$$x_n = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{26}$$

From the equation (23), it can be seen that the computation of X[k] requires N^2 complex multiplication. Thus, the DFT is N^2 process. The algorithm was developed by Turkey Cooley in 1965 called the fast Fourier Transformation (FFT) and speeds up the process by computing the DFT using O($N * logN$) operations (Turkey and Cooley, 1965) [2]. FFT is a faster algorithm for computing the DFT. The FFT is simply an efficient method of computing the DFT and it also reduces round round off of error by factor of $logN/N$, where N is number of data samples (Turkey and Cooley, 1965; Hsu, 1995; Orfanidis, 2010).

3. Preliminaries

3.1. Property of theTrigonometric System

Theorem 3.1 The trigonometric system (9) is orthogonal on the interval $-\pi \le x \le \pi$ (also on $0 \le x \le 2\pi$ or any other interval of length 2π because of periodicity); that is, the integral of the product of any two functions in (9) over that interval is 0, so that for any integers n and m, we have

$$\int_{-\pi}^{\pi} \cos nx\cos mx dx = 0, n \ne m \tag{27}$$

$$\int_{-\pi}^{\pi} \sin nx\sin mx dx = 0, n \ne m \tag{28}$$

$$\int_{-\pi}^{\pi} \cos nx\sin mx dx = 0, n \ne m \tag{29}$$

From Period 2π To Any Period $p = 2L$

Periodic functions in applications may have any period, not just 2π as in the above disscuse (chosen to have simple formulas). The notation $p = 2L$ for the period is practical because L will be a length of a violin string of a rod in heat conduction. The transition from period 2π to be period $p = 2L$ is effected by a suitable change of scale, presented as follows.

Let $f(x)$ have period $p = 2L$. Then introduce a new variable v such that $f(x)$, as a function of v, has period 2π set $x = \frac{p}{2\pi} v$ so that $v = \frac{2\pi}{p} x = \frac{\pi}{L} x$, $v = \pm\pi$ corresponding to $x = \pm L$ this mean that as a function of v, has period 2π and therefore a Fourier series of the form

$$f(x) = f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\propto} (a_n \cos nv + b_n \sin nv) \tag{30}$$

With coefficient obtaind by Euler forumala

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) dv \tag{31}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \cos nv\, dv \tag{32}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \sin nv\, dv \tag{33}$$

It could be used these formula directly, but the change variable to x simplifies the calculation. Since

$$v = \frac{\pi}{L} x \text{ thus } dv = \frac{\pi}{L} dx$$

and integrate over x from $-L$ to L. Consequently, obtain for a function $f(x)$ of period $2L$ the Fourier series.

$$f(x) = a_0 + \sum_{n=1}^{\propto} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x\right) \tag{34}$$

With fourier coefficient

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \tag{35}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\, dx,\ n = 0, 2, \ldots \tag{36}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x)\, s\, in \frac{n\pi}{L} x\, dx, n = 1, 2, \ldots \tag{37}$$

Periodic functions with periods 2L are more realistic. Equations (34) is thus more practical in engineering analysis.

3.2. Convergence of Fourier Series

Definition: (Piecewise smooth functions). For f defined on $[a, b]$, f is piecewise smooth on $[a, b]$ if there is a partition of $[a, b]$

$a = x_0 < x_1 < x_2 < \cdots < x_p = b$ Such that f is continuously differentiable on each subinterval (x_j, x_{j+1}), and at each x_j, f or its derivative f' has at most a jump discontinuity.

Theorem 3.2: (Dirichlet Conditions)

Suppose that

i) $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$

ii) $f(x)$ is periodic outside $(-L, L)$ with period 2L

iii) $f(x)$ and $f'(x)$ are piecewise smooth on $(-L, L)$

Then the series (34) with Fourier coefficients converges to

a. $f(x)$ if x is a point of continuity

b. $\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

3.3. Fourier Series for Non-periodic Functions

Assume that it is interested in approximating the function only over a limited interval and do not care whether the approximation holds outside of that interval. Suppose that a function defined for all x-values, but only interested in representing it over (0, L). Because we will ignore the behavior of the function outside of (0, L). We can be redefine the behavior outside that interval and right to show two possible redefinitions.

In the first redefinition, reflected the portion of $f\ (x)$ about the y-axis and have extended it as a periodic function of period 2L. This creates an even periodic function

$$f\ (x) \text{ is even if } f\ (-x) = f\ (x).$$

And also if reflect it about the origin and extend it periodically, it can be create an odd periodic function of period 2L.

$$f\ (x)\ is \text{ odd if } f\ (-x) = -f\ (x).$$

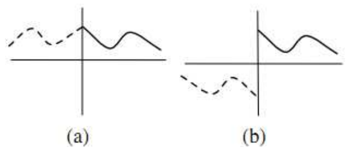

Figure 1. *Even (a) and odd (b) Functions.*

There are two important relationships for integrals of even and odd functions.

$$\text{If } f\ (x) \text{ is even then } \int_{-L}^{L} f(x)dx = 2\int_0^L f(x)\, dx \quad (38)$$

$$\text{If } f\ (x) \text{ is odd, then } \int_{-L}^{L} f(x)dx = 0. \quad (39)$$

Properties Of Even And Odd Function

1. The product of two even functions is even, if $f\ (x)$ is even, then $f\ (x)cos(nx)$ is even
2. The product of two odd functions is even, if $f\ (x)$ is odd, then $f\ (x)sin(nx)$ is even
3. The product of an even and an odd function is odd;
   if $f\ (x)$ is even, then $f\ (x)sin(nx)$ is odd
   if $f\ (x)$ is odd, then $f\ (x)cos(nx)$ is odd
4. The Fourier series expansion of an odd function will contain only sine terms and all the $a_n$coefficients are zero.
5. The Fourier series expansion of an even function will contain only cosine terms and all the $b_n$coefficients are zero.

3.4. Complex Fourier Series

An alternative, and often more convenient, approach to Fourier series is to use complex exponentials instead of

$$f(x) \approx A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx) \quad (40)$$

Euler's formula

$$e^{ikx} = \cos kx + i \sin kx,$$

$$e^{-ikx} = \cos kx - i \sin kx,$$

Shows that how to write the trigonometric functions

$$\cos kx = \frac{e^{ikx}+e^{-ikx}}{2}, \sin kx = \frac{e^{ikx}-e^{-ikx}}{2i},$$

Where i is the imaginary unit with the property that $i^2 = -1$. Equation (40) can be written as

$$f(x) = \sum_{k=0}^{\infty}(c_k e^{ikx} + c_{-k}e^{ikx}) = 2c_0 + \sum_{k=1}^{\infty}(c_k + c_{-k})\cos(kx) + i(c_k + c_{-k})\sin(kx) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad (41)$$

We can match up the A's and B's of equation (40) to c's of equation (41)

$$a_n = c_k + c_{-k}\ b_n = i(c_k - c_{-k})$$

$$c_k = \frac{a_k - ib_k}{2}\ c_{-k} = \frac{a_k + ib_k}{2i}$$

For integer k and j, it is true that

$$\int_0^{2\pi}(e^{ikx})(e^{ijx}) = \int_0^{2\pi} e^{i(k+j)x} = \begin{cases} 0\ for\ k \neq -j \\ 2\pi\ for\ k = -j \end{cases}$$

This allows us to evaluate c_k by the following. For each fixed j we get

$$f(x)e^{-ijx} = \sum_{-\infty}^{\infty} c_j e^{i(j-k)x}$$

$$\int_0^{2\pi} f(x)e^{-ijx} = 2\pi c_j$$

$$c_j = \frac{1}{2\pi} f(x)e^{-ijx}\ j = 0, \pm1, \pm2, \ldots$$

3.5. Partial Differential Equation

Consider a second order partial differential equation given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0 \quad (42)$$

Where A, B, C, D, E, F and G are functions of x and y or real constants. The partial differential equation (42) is called

a) Elliptic equation if $B^2 - 4AC < 0$

b) Parabolic equation if $B^2 - 4AC = 0$

c) Hyperbolic equation if $B^2 - 4AC > 0$

Examples: 3.1

Parabolic equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ One dimensional heat equation}$$

Hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ One dimensional wave equation}$$

Elliptic equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ Two dimensional Laplace equations}$$

3.6. The Heat Conduction Equation

Conservation of heat can be used to develop a heat balance for the differential element in the long, thin insulated rod shown in Figure 2 below. However, rather than examine the steady State case, the present balance also considers the amount of heat stored in the element over a unit time period Δt.

In a metal rod with non-uniform temperature, heat (thermal energy) transferred from regions of higher temperature to regions of lower temperature. Three physical principles are used here.

i. Heat (or thermal) energy of a body with uniform properties:

$$\text{Heat energy} = cmu \quad (43)$$

Where m is the body mass, u is the temperature, c is the specific heat, units [c] = $L^2T^{-2}U^{-1}$

Basic units are M mass, L length, T time, U temperature and C is the energy required to raise a unit mass of the substance 1 unit in temperature.

ii. Fourier's law of heat transfer:

The rate of heat transfer proportional to negative temperature gradient,

$$\frac{Rate\ of\ heat\ tranfer}{area} = -K_0 \frac{\partial u}{\partial x} \quad (44)$$

Where K_0 is the thermal conductivity, units $[K_0]$ = $MLT^{-3}U^{-1}$. In other words, heat transferred from areas of high temperature to low temperature.

iii. Conservation of energy. Consider a uniform rod of length L with non-uniform temperature lying on the x-axis from $x = 0\ to\ x = L$. By uniform rod, we mean the density ρ, specific heat c thermal conductivity K_0, cross-sectional area A are all constant. Assume the sides of the rod are insulated and only the ends may be exposed. Also assume there is no heat source within the rod. Consider an arbitrary thin slice of the rod of width Δx between x and $x + \Delta x$. The slice is so thin that the temperature throughout the slice is $u\,(x, t)$.

Thus,

$$\text{Heat energy of segment} = c \times \rho A \Delta x \times u = c\rho A \Delta x u\,(x, t) \quad (45)$$

By conservation of energy,

Change of Heat in from Heat out from

Heat energy = $-cmu$

of segment in time Δt left boundary right boundary

From Fourier's Law (1),

$$c\rho A\Delta x u\,(x, t + \Delta t) - c\rho A \Delta x u\,(x, t) = \Delta t A \left(-K_0 \frac{\partial u}{\partial x}\right)_x - \Delta t A \left(-K_0 \frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

Rearranging yields (recall ρ, c, A, K_0 are constant),

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{K_0}{\rho c}\left(\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right)$$

Taking the limit $\Delta t, \Delta x \to 0$ gives the Heat Equation,

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \quad (46)$$

Where, $\kappa = \frac{K_0}{\rho c}$

(45) is the heat conduction equation in one dimension. The variable u(x, t) is the temperature at position x and time t. This equation is an example of parabolic equation. Depending on the equation we must consider changes in time as well as in space.

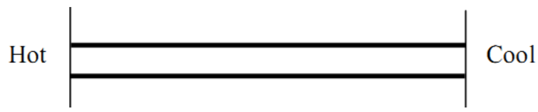

Figure 2. *A thin rod, insulated at all points except at its end.*

3.7. Schemes for Heat Equation

The finite difference method is one of several techniques for obtaining numerical solutions to Equation (45). In all numerical solutions the continuous partial differential equation (PDE) is replaced with a discrete approximation. The discrete approximation results in a set of algebraic equations that are evaluated for the values of the discrete unknowns [20]. The mesh is the set of locations where the discrete solution is computed. These points are called nodes, and if one were to draw lines between adjacent nodes in the domain the resulting image would resemble a net or mesh.

Two key parameters of the mesh are Δx, the local distance between adjacent points in space, an Δt, the local distance between adjacent time steps. Applying the finite-difference method to a differential equation involves replacing all derivatives with difference formulas [18]. In the heat equation there are derivatives with respect to time, and derivatives with respect to space. Using different combinations of mesh points in the difference formulas results in different schemes. In the limit as the mesh spacing (Δx and Δt) go to zero the numerical solution obtained with any useful scheme will approach the true solution to the original differential equation.

3.7.1. Forward Time, Centered Space

The forward time centered space (FTCS) method sometimes called Schmidt method. The base point for the finite difference approximation (FDA) of the partial differential equation (PDE) is grid point (i, n). The finite difference equation (FDE) approximates the partial derivative u_t by the first order forward time approximation equation, and the partial derivative u_{xx} by the second order centered space [20]. Approximate the time derivative in heat equation with forward difference

$$\frac{\partial u}{\partial t}|_{t_{m+1},x_i} = \frac{{u_i}^{m+1}-{u_i}^{m}}{\Delta t} + O(\Delta t) \tag{47}$$

Notation $O(\Delta t)$ in (47) is used to express the dependence of the truncation error on the time spacing.

Use the central difference approximation and evaluate all terms at time m.

$$\frac{\partial^2 u}{\partial x^2}|_{x_i} = \frac{{u_{i-1}}^{m}-2{u_i}^{m}+{u_{i+1}}^{m}}{\Delta \mathrm{x}^2} + O(\Delta \mathrm{x}^2) \tag{48}$$

Where the notation $O(\Delta t)$ in (48) is used to express the dependence of the truncation error on the mesh space.

Substitute Equation (47) into the left hand side of equation (45); substitute Equation (46) into the right hand side of heat equation; and collect the truncation error terms to get

$$\frac{{u_i}^{m+1}-{u_i}^{m}}{\Delta t} = c\,\frac{{u_{i-1}}^{m}-2{u_i}^{m}+{u_{i+1}}^{m}}{\Delta \mathrm{x}^2} + O(\Delta t) + O(\Delta \mathrm{x}^2) \tag{49}$$

The temporal errors and the spatial errors have different orders. Also notice that we can explicitly solve for ${u_i}^{m+1}$ in terms of the other values of u. Drop the truncation error terms from equation (49) and solve for ${u_i}^{m+1}$ to get.

$${u_i}^{m+1} = {u_i}^{m} + \frac{c\Delta t}{\Delta \mathrm{x}^2}({u_{i-1}}^{m} - 2{u_i}^{m} + {u_{i+1}}^{m}) \tag{50}$$

Equation (50) is called the Forward Time, Centered Space or FTCS approximation to the heat equation, reader may be refer on (Gerald, 2011).

3.7.2. Backward Time Centered Space

In this subsection the one dimensional diffusion equation is solved by backward time centered space method. This method is also called the fully implicit method [12]. The finite difference equation which approximates the partial differential equation is obtained by replacing the exact partial derivative u_t by the first order backward time approximation, which is developed below, and the exact partial derivative u_{xx} by the second order centered space. In the derivation of Equation (50) the forward difference was used to approximate the time derivative on the left hand side of equation (45). Now, choose the backward difference,

$$\frac{\partial u}{\partial t}|_{t_{m+1},x_i} = \frac{{u_i}^{m}-{u_i}^{m-1}}{\Delta t} + O(\Delta t) \tag{51}$$

Substitute equation (50) into the left hand side of equation (45); substitute equation (46) into the right hand side of heat equation; and collect the truncation error terms to get

$$\frac{{u_j}^{k+1}-{u_j}^{k}}{\Delta x^2} = \frac{{u_{j-1}}^{k+1}-2{u_j}^{k+1}+{u_{j+1}}^{k+1}}{\Delta x^2} + O(\Delta t) + O(\Delta \mathrm{x}^2) \tag{52}$$

Implementation of the BTCS scheme requires solving a system of equations at each time step. In addition to the complication of developing the code, the computational effort per time step for the BTCS scheme is greater than the computational effort per time step of the FTCS scheme. The BTCS scheme has one huge advantage over the FTCS Gerald (2011).

4. Approximation of Functions by Fourier Series

In this chapter we study the function approximation by Fourier series. We will discuss square wave function, fourier exponential and trigonemetric polynomial. And we use DFT to approximate function value from data set [23]. We will also discuss the accuracy of Fourier approximation of functions and error in approximation. The approximate function are represented graphically to compare with the original function.

4.1. Approximating the Square Wave Function using Fourier Sine Series

The Square Wave function is commonly called a step function which alternates between two function values periodically and instantaneously. In particularly the square wave function graphed from x = −1 to x = 1 is presented in the Figure 24. By summing sine waves it is possible to replicate the square wave function almost exactly, however, there is a discontinuity in this periodic function, meaning the Fourier series will also have a discontinuity. It is clear in Figure 24 that the discontinuity will appear at x = 0, where the functions jump from −1 to 1, the equation of this function is presented in equation (53).

$$f(x) = \begin{cases} -1\ for -1 \le x < 0 \\ \ \ 1\ for\ 0 \le x \le 1 \end{cases} \tag{53}$$

The square wave is the $2L-$periodic extension of the function $f(x)$.

Since $f(x)$ is odd periodic extension of the function, Fourier series expansions of $f\ (x)$ on [-1, 1] that has only sine terms. We get b_n for the odd extension of $f\ (x)$ on [-1, 1]

$$b_n = \frac{2}{2L}\int_{-L}^{L} f(x)\sin(n\pi x)\,dx = \frac{2}{2}\int_{-1}^{0} -\sin(n\pi x)\,dx + \frac{2}{2}\int_{0}^{1}\sin(n\pi x)\,dx = \begin{cases}\frac{4}{n\pi}\ if\ n\ is\ odd\\ 0\ if\ n\ is\ even\end{cases}$$

Since $a_0 = 0, a_n = 0$ and $b_n = \frac{4}{n\pi}$, now it can be represented as a Fourier sine series

$$f(x) \approx \sum_{nodd}^{M} b_n \sin(nx) = \frac{4}{\pi}\sum_{n=0}^{M}\frac{\sin(2n+1)x}{2n+1} \quad (54)$$

In which form the cosine terms have been automatically dropped. Equation (54) represents Fourier sine series. The Fourier sum of sine can be used to accurately approximate the square wave function. In equation (54), n represents the number of coefficients. Numerical values of square wave function for $n = 1, 5, 30, 100$ and 1000 are given in table 1. MATLAB Code given in Appendix A is used to plot the square wave function and Figure 2, 3, 4, 5, and 6 show square wave function for $n = 1$, 5, 30 1000 and respectively.

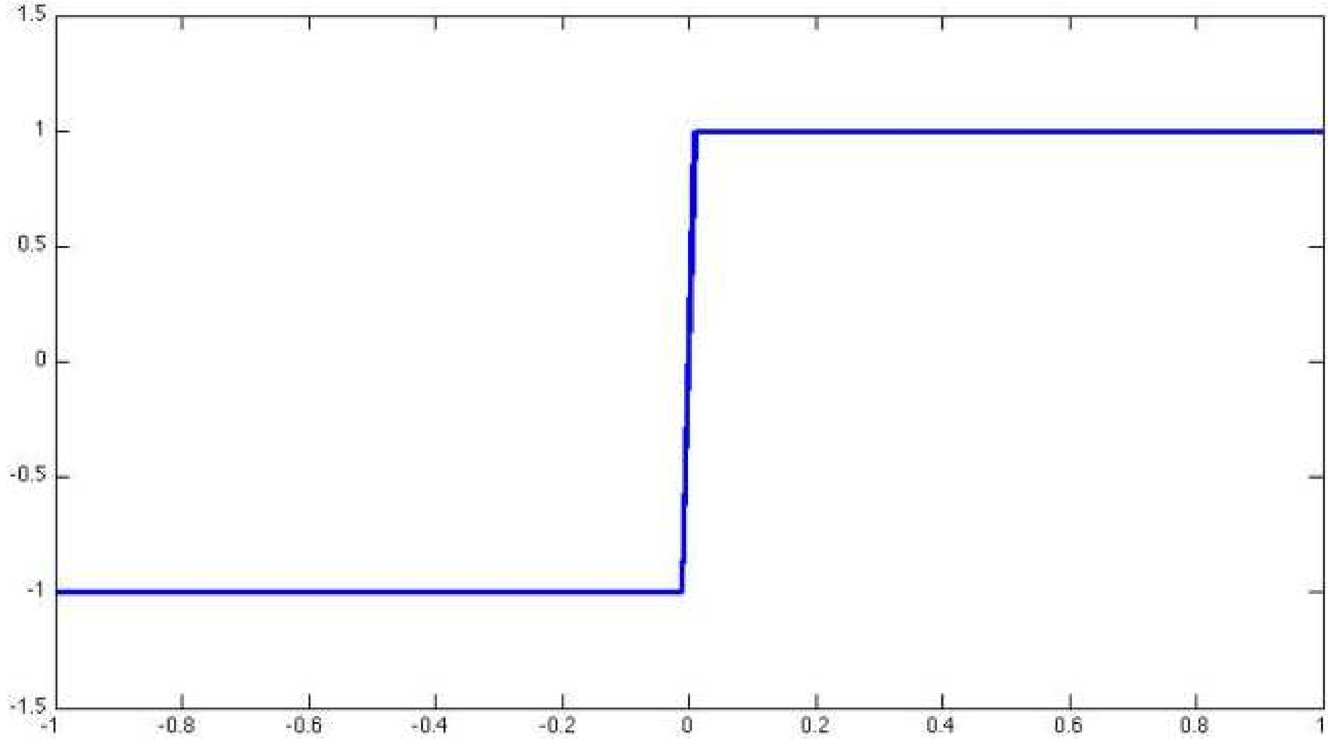

Figure 3. *Square wave function.*

Table 1. *Numerical values of sine wave.*

X	S_1	Error	S_5	Error	S_{30}	Error	S_{100}	Error	S_{1000}	Error
-1	-0.00	1.00	-0.00	1.00	-0.00	1.00	-0.00	1.00	-0.00	1.00
-0.8	-0.7484	0.2516	-1.1520	0.1520	-0.9641	0.0359	-0.9892	0.0108	-0.9989	0.0011
-0.6	-1.2109	0.2109	-0.9615	0.0385	-0.9777	0.0223	-0.9933	0.0067	-0.9993	0.0007
-0.4	-1.2109	0.2109	-0.9615	0.0385	-0.977	0.0223	-0.9933	0.0067	-0.9993	0.0007
-0.2	-0.7484	0.2516	-1.1520	0.1520	-0.9641	0.0359	-0.9892	0.0108	-0.9989	0.0011
0.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.2	0.7484	0.2516	1.1520	0.1520	0.9641	0.0359	0.9892	0.0108	0.9989	0.0011
0.4	1.2109	0.2109	0.9615	0.0385	0.977	0.0223	0.9933	0.0067	0.9993	0.0007
0.6	1.2109	0.2109	0.9615	0.0385	0.9777	0.0223	0.9933	0.0067	0.9993	0.0007
0.8	0.7484	0.2516	1.1520	0.1520	0.9641	0.0359	0.9892	0.0108	0.9989	0.0011
1	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.0000	0.000	1.000

S_1= partial sum with 1 term, S_5= partial sum with 5 terms, S_{30}= partial sum with 30 terms, S_{100}= partial sum with 100 terms and S_{1000}= partial sum with 1000 terms.

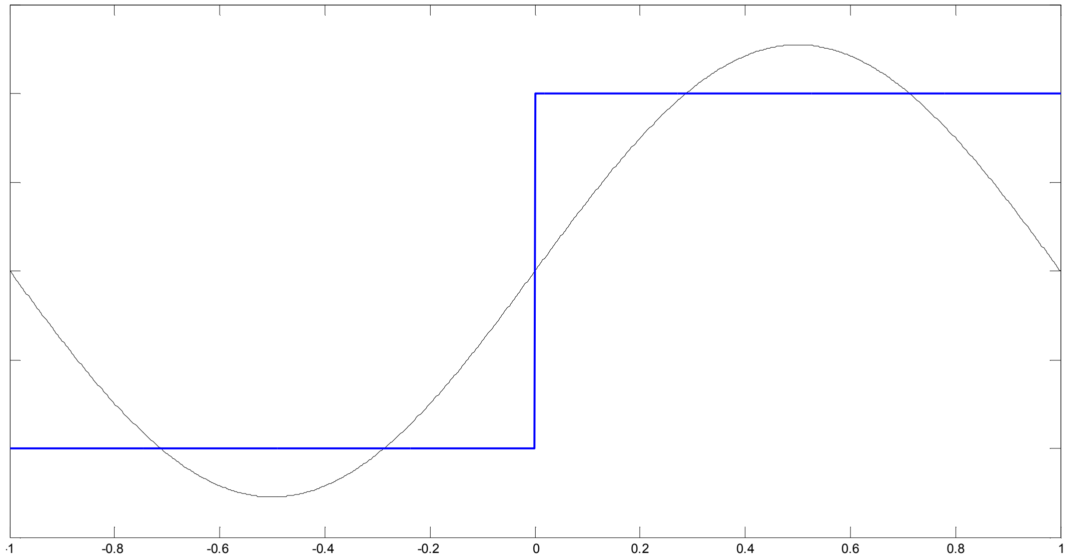

Figure 4. *Sine wave function with n=1 coeff.*

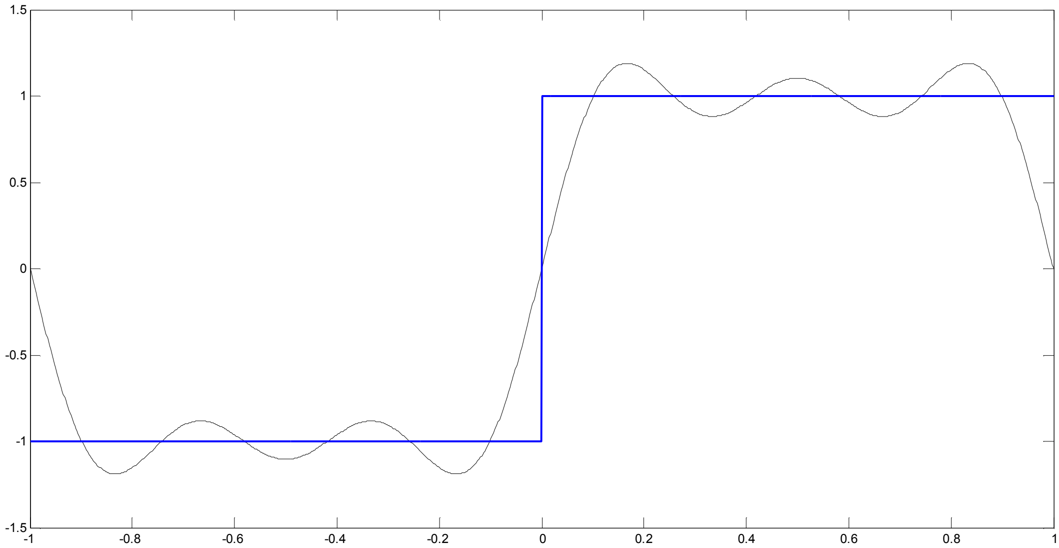

Figure 5. *Sine wave function n=5 coeff.*

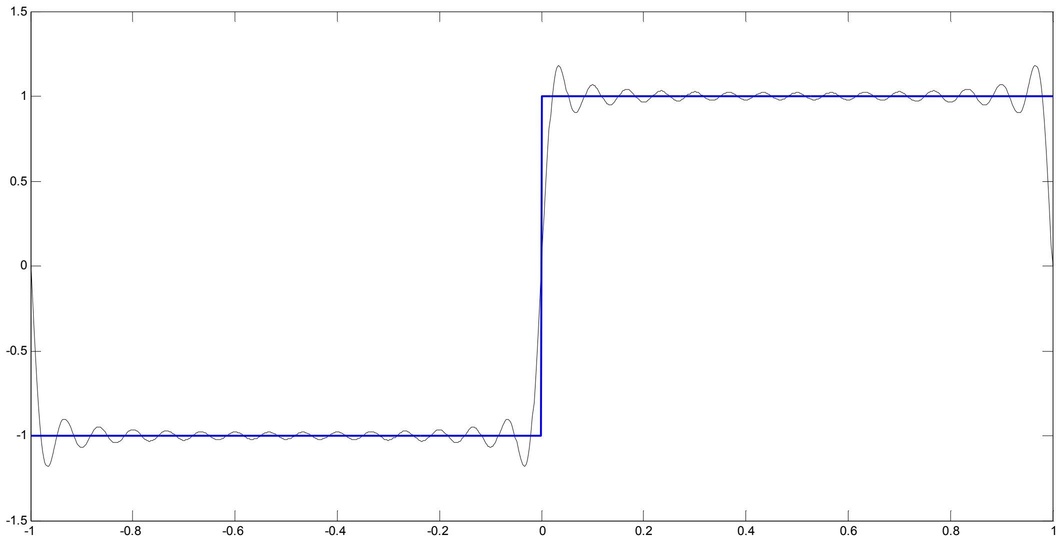

Figure 6. *Sine wave function, n=30 coeff.*

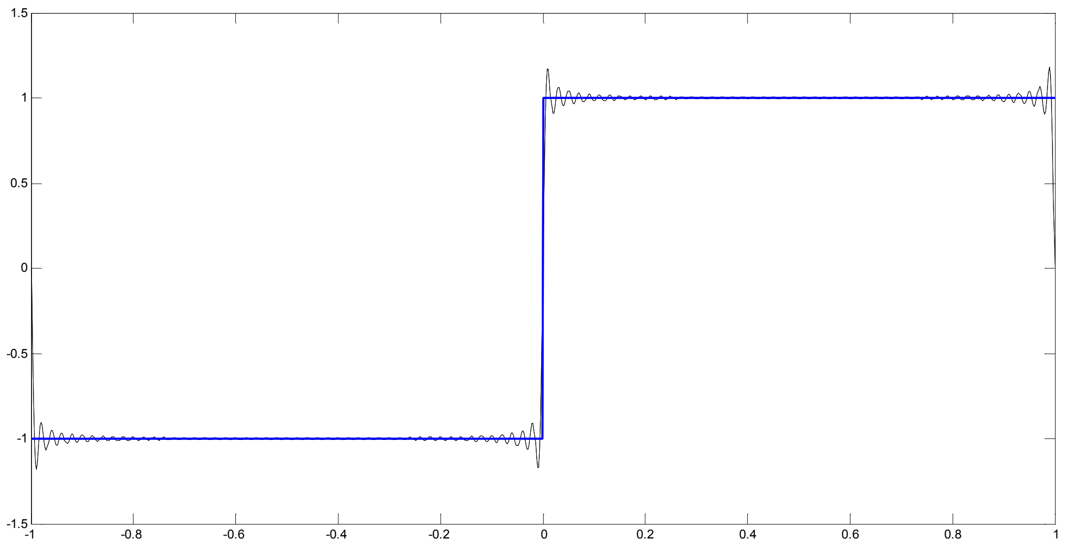

Figure 7. *Sine wave function, n=100 coeff.*

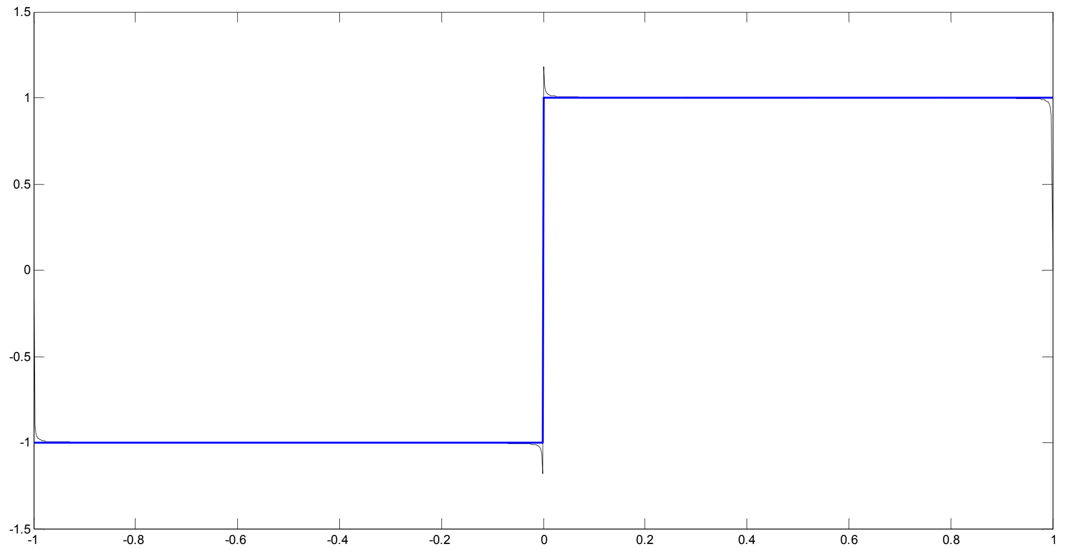

Figure 8. *Sine wave function with n=1000 coeff.*

Discussions:

We compare the function given in equation (54) with actual function by changing the number of coefficients n and setting the number of points $N = 1000$. We start with $n = 1, n = 5, n = 30, n = 100$ and $n = 1000$ and compare with the actual function as shown in the Figures 4, 5, 6, 7 and 8 respectively. From the figures we see that as the number of n increase, the graph of function (54) comes closer to the actual function For $n = 1000$ the graph looking more and more like a square wave, which is seen in Figure 3, but we see that the persistent overshoot at the end with a large number of terms, the fit is very good.

There is visible error at the points: $x = 1, x = -1$ and $x = 0$, i.e. where the function is discontinuous. From the Figures above, it is clear that the accuracy increased as more and more coefficients are used. And the error appears to be minimal as more coefficients are used. This occurrence is referred to as the Gibb's Phenomenon. J. Willard Gibbs discovered that there will always be an overshoot at the points of discontinuity when using Fourier series approximation.

Thus we observe that the approximation function get closer to the actual function on the given interval. From the Table 4 we also observe that at the end points and at mid point the approximation function does not converge to actual function $f(x)$. Since $f(x)$ is $2l$ periodic function and has the same value at the end, which is 0.

4.2. Fourier Series Approximation of Line

Let consider the equation of the line represented in (55),

$$f(x) = \frac{1}{2}(\pi - x) \text{ On } 0 \leq x \leq 2\pi \tag{55}$$

Fourier series expansion of function $f(x)$ in (55) is given by

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \, b_n \sin nx$ Where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx, a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx \; n = 0, 1, \ldots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx\ n = 1,2,3,\ldots.$$

By integrating we obtained Fourier coefficients

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}\pi\cos nx\,dx - \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}x\cos nx\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\pi\sin n\pi}{n} - \frac{\pi\sin 0}{n}\right] - \frac{1}{2\pi}\left[\frac{\pi\sin n\pi}{n} - \frac{0\sin 0}{n} + \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}\pi\sin nx\,dx - \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}x\sin nx\,dx = 0 + \frac{1}{2\pi}\left[\frac{x\cos nx}{n}\right]\Big|_{x=0}^{x=2\pi} + \frac{\sin nx}{n^2}\Big|_{x=0}^{x=2\pi} = \frac{1}{n}$$

Thus, this yeild Fourier sine series of

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n}\sin nx \approx F(x) = \sum_{n=1}^{M}\frac{1}{n}\sin nx \qquad (56)$$

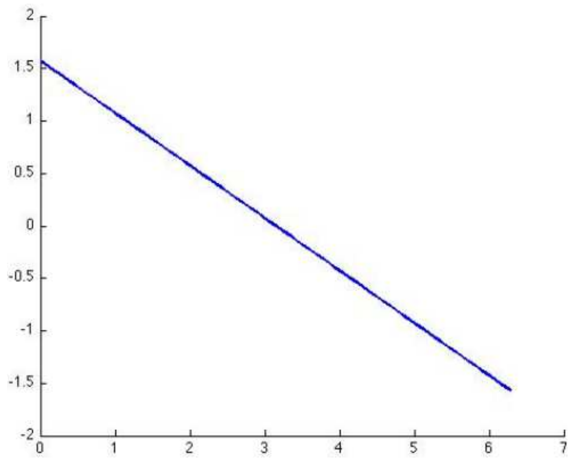

Figure 9. *Graph of* $f(x) = x$.

Numerical evaluation and error of the above Fourier sine series is give in Table 5. MATLAB codes that are given in appendix B and C are used to find numeiracal solution and to draw the graph of line function together with Fourier sine series (55) and Figures 10, 11, 12, 13 and 13 show the comparisom of graph of approximation with actual function

Table 2. *Numerical solution of line approximation by square wave function.*

X	S_1	Error	S_{10}	Error	S_{50}	Error	S_{100}	Error	S_{1000}
0	0.00	1.5708	0.00	1.5708	0.00	1.5708	0.00	1.5708	0.00
0.2π	0.6428	0.5789	1.1399	0.0819	1.2444	0.0226	1.2143	0.0074	1.2210
0.6π	0.9848	0.1121	0.9059	0.0332	0.8698	0.0029	0.8765	0.0038	0.8731
0.8π	0.8660	0.3424	0.5784	0.0548	0.5179	0.0057	0.5293	0.0057	0.5242
π	0.3420	0.1675	0.1990	0.0245	0.1840	0.0094	0.1771	0.0025	0.1748
1.2π	-0.3420	0.1675	-0.1990	0.0245	-0.1840	0.0094	-0.1771	0.0025	-0.1748
1.4π	-0.8660	0.3424	-0.5784	0.0548	-0.5179	0.0057	-0.5293	0.0057	-0.5242
1.6π	-0.9848	0.1121	-0.9059	0.0332	-0.8698	0.0029	-0.8765	0.0038	-0.8731
1.8π	-0.6428	0.5789	-1.1399	0.0819	-1.2444	0.0226	-1.2143	0.0074	-1.2210
2π	-0.00	1.5708	-0.00	1.5708	-0.00	1.5708	-0.00	1.5708	-0.00

S_1= partial sum with 1 term, S_{10}= partial sum with 10 terms, S_{50}= partial sum with 50 terms, S_{100}= partial sum with 100 terms and S_{1000}= partial sum with 1000 terms.

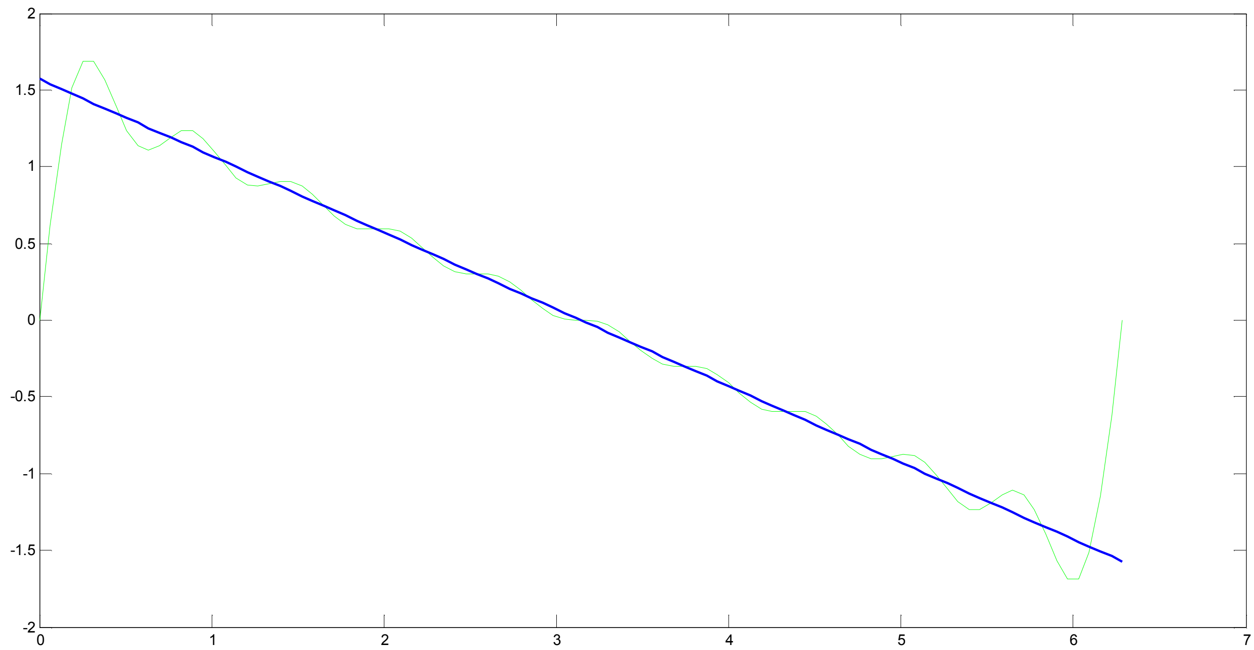

Figure 10. *Line approx with n=1 coeff.*

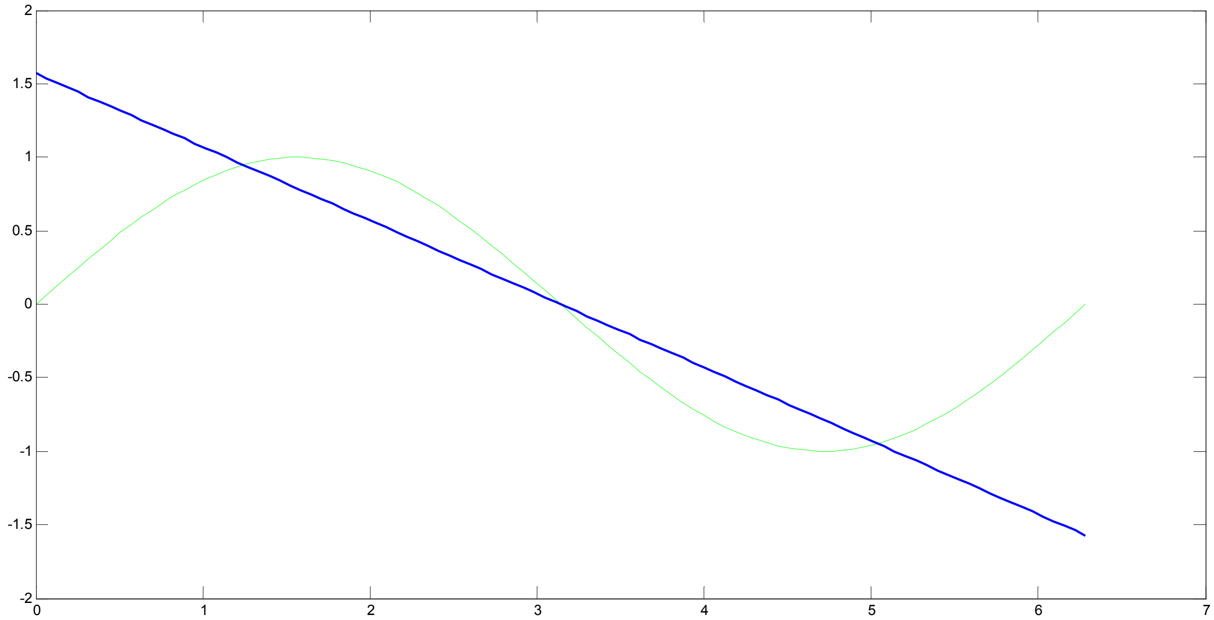

Figure 11. *Line approx with n=10 coeff.*

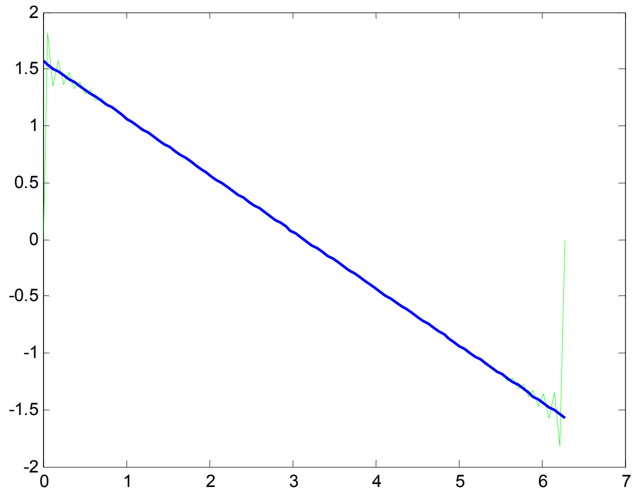

Figure 12. *Line approx with n=50 coeff.*

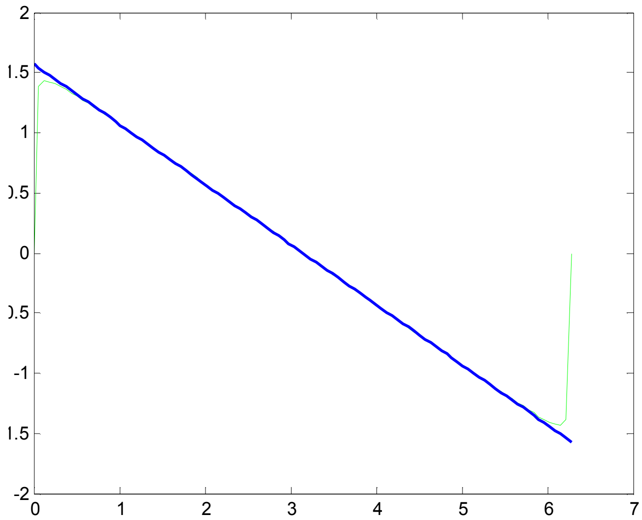

Figure 13. *Line approx with n=100 coeff.*

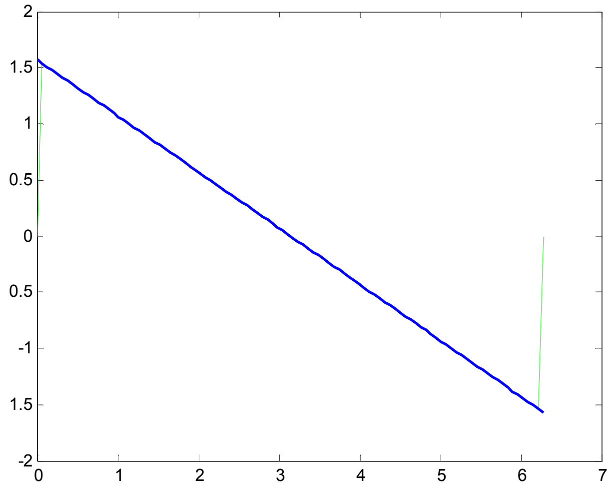

Figure 14. *Line approx with n=1000 coeff.*

Discussions:

From the Figure 8, Figure 9, and Figure 10, we see that as n increases the approximation function graph comes very close to the line graph, but there is discontinuity at the end points. The reason for this discontinuity is not the same as in the first case. The function here is continuous and bounded, but not periodic. Due to the sine function being periodic, it cannot approximate a non periodic function with complete accuracy. There is visible error at the end which is never be disappeared due to Gibb' phenomena.

4.3. Approximation of a Line by Fourier Exponential Approximation

In the section 4.2 have seen that line is approximated by sine wave function, now we have approximated another line which is not a sum of sine waves, but instead a sum of complex exponential functions. The equation of the line over the interval $[-1,1]$ is

$$h(x) = x \tag{57}$$

The function used to approximate the line have been presented by imaginary exponential as

$$h(x) \approx F(x) = \sum_{-N}^{N} f(x)\, e^{i\pi kx},\ k = 0, 1, \ldots \tag{58}$$

Where $f(x)$ is coefficient of exponential in (58) and given by the expression

$$f(x) = \frac{1}{2l}\int_{-l}^{l} F(x)\, e^{-i\pi kx} \tag{59}$$

MATLAB Codes to represent the graph of the function with different number of coefficient are given in appendix D and E. MATLAB Codes in appendix D is created to calculate the Coefficients of Exponential and MATLAB Codes in appendix E is to reconstruction the function and we are approximating using the coefficients from the earlier code. Our code uses NC to represent the number of coefficients used. By increasing it we can better approximate function. The graph of the function we are approximating and the exponential function used to approximate with NC = 1 shown Figure 13, NC = 5 Figure 14: NC = 10 Figure 15, NC = 25 Figure 16 and NC=30 Figure 17 that are showed in the following figures.

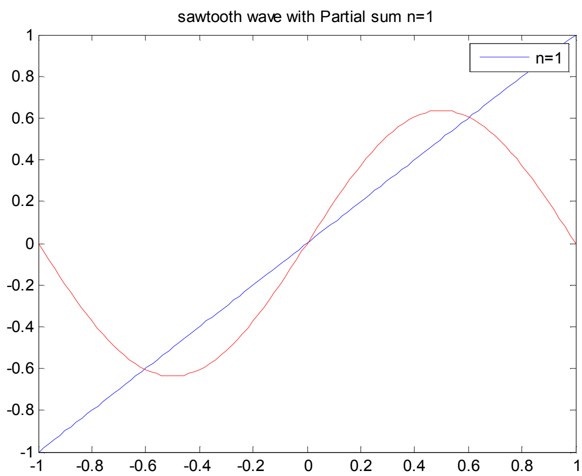

Figure 15. *Line approx by Exp NC=1.*

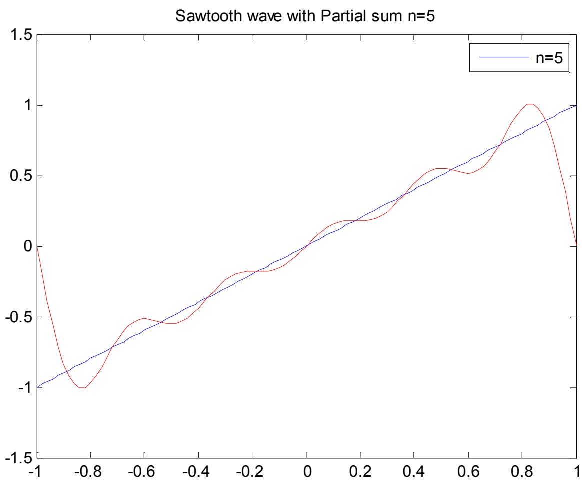

Figure 16. *Line approx by Exp NC=5.*

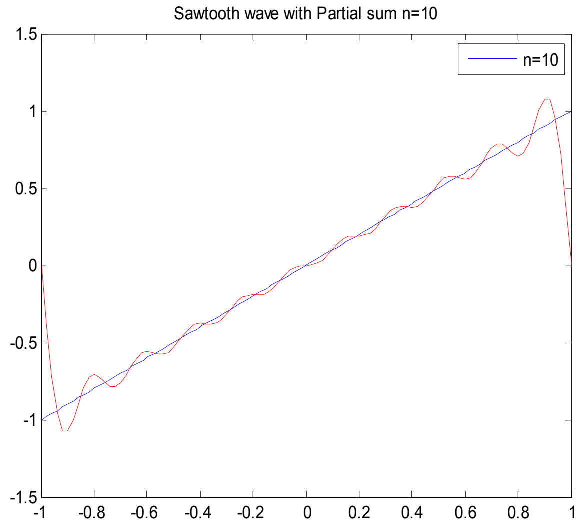

Figure 17. *Line approx by Exp NC=10.*

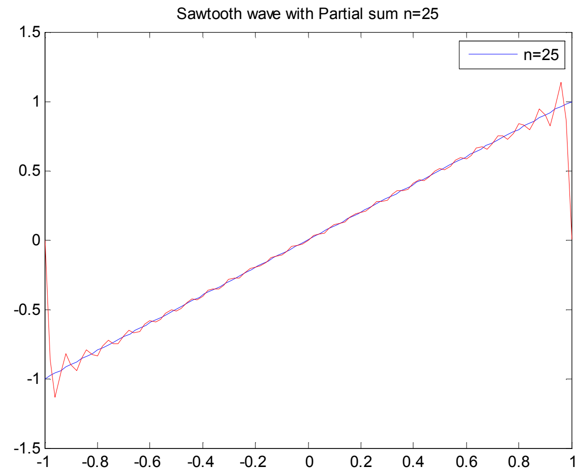

Figure 18. *Line approx by Exp NC=25.*

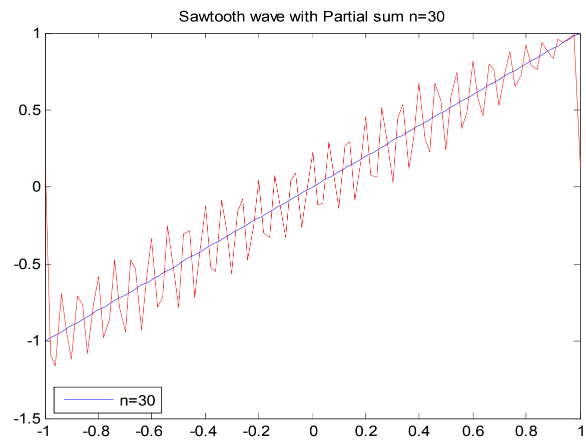

Figure 19. *Line approx by Exp NC=30.*

Discusion:

It can seen that even with an exponential function used for approximation, the Gibbs phenomenon persists along the boundaries which error visible at the boundary. It can be seen that as the number of coefficients increase the approximation obtained better except that in figure 4.17, the approximation appears to be worse. This would be because, with the quad function we used in MATLAB to integrate the function we see a problem as the number of coefficients exceeds 25.

4.4. Approximation by Trigonometric Polynomial and Least Square Error Evaluation

Fourier series play a prominent role not only in differential equations but also in approximation theory, an area that is concerned with approximating functions by other functions usually simpler functions. Here is how Fourier series come into the picture.

Let $f(x)$ be a function on the interval $-\pi \leq x \leq \pi$ that can be represented on this interval by a Fourier series. The Nth partial sum of the Fourier series.

$$f(x) \approx a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \text{ is the approximation of } f(x) \tag{60}$$

In equation (60) we can choose an arbitrary N and keep it fixed. Then approximation of f_N by a trigonometric polynomial of the same degree N, that is represented by a function of the form.

$$f_N(x) = A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx) \tag{61}$$

Error in in the approximation of $f_N(x)$ is given by

$$\mathrm{E}(a_0, a_1, a_2, \dots, a_N, b_1, \dots, b_N) = \int_{-\pi}^{\pi}(f(x) - f_N(x))^2\, dx$$

called the square error of f_N relative to the function $f(x)$ on the interval $-\pi \leq x \leq \pi$

Where $a_0, a_1, a_2, \dots, a_N, b_1, \dots, b_N$ are coefficients of trigonometric polynomial which is determining by least square condition. N being fixed, it required to determine the coefficient in (60) such that E is minimum.

Since, $(f - f_N)^2 = f^2 - 2ff_N + {f_N}^2$ it follows that

$$\mathrm{E}(a_0, a_1, a_2, \dots, a_N, b_1, \dots, b_N) = \int_{-\pi}^{\pi}(f(x) - f_N(x))^2\, dx = \int_{-\pi}^{\pi} f^2 dx - 2\int_{-\pi}^{\pi} ff_N\, dx + \int_{-\pi}^{\pi} {f_N}^2 dx$$

intgrating, we have

$$\int_{-\pi}^{\pi} {f_N}^2 dx = \int_{-\pi}^{\pi}\left(A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx)\right)^2 \mathrm{dx} = \int_{-\pi}^{\pi}\left({A_0}^2 + \sum_{n=1}^{N} 2A_0(A_n \cos nx + B_n \sin nx) + (A_n \cos nx + B_n \sin nx)^2\right)\mathrm{dx}$$

$$= 2\pi {A_0}^2 + 2A_0\int_{-\pi}^{\pi}(A_1 cosx + A_2 cos2x + \cdots + A_N cosNx + B_1 sinx + B_2 sin2x + \cdots + B_N sinNx)dx + \int_{-\pi}^{\pi}\left({A_1}^2 cos^2x + {A_2}^2 cos^2 2x + \cdots + {A_N}^2 cos^2 Nx + {B_1}^2 sin^2 x + {B_1}^2 sin^2 2x \dots. + {B_N}^2 sin^2 Nx\right)dx + 2\int_{-\pi}^{\pi}\sum_{n=1}^{N} A_n B_n cosnxsin\, nx \mathrm{dx} \tag{62}$$

Thus the integral of $cos^2 nx$ and $sin^2 nx$ equal to π and the integral of $cosnx, \sin nx$ and $(cosnx)(\sin nx)$ which equal to zero. Now equation (61) becomes

$$\int_{-\pi}^{\pi} {f_N}^2 dx = \pi(2{A_0}^2 + {A_1}^2 + {A_2}^2 + \cdots + {A_N}^2 + {B_1}^2 + {B_1}^2 \dots. + {B_N}^2) \tag{63}$$

Using (60) and by orthogonallity property of the integral we have

$$\int_{-\pi}^{\pi} \cos nx cos\, mx dx = 0 = \int_{-\pi}^{\pi} \sin nx sin\, mx dx m = \int_{-\pi}^{\pi} \cos nx sin\, mx dx, n \neq m$$

Hence

$$\int_{-\pi}^{\pi} ff_N dx = \pi(2A_0 a_n + \sum_{n=1}^{N}(A_n a_n + B_n b_n)) \tag{64}$$

Now expresion (61) reduce to

$$\mathrm{E} = \int_{-\pi}^{\pi} f^2 dx - \pi 2[2A_0 a_n + \sum_{n=1}^{N}(A_n a_n + B_n b_n)] + \pi[2{A_0}^2 + \sum_{n=1}^{N}({A_n}^2 + {B_n}^2)] \tag{65}$$

Now take $A_n = a_n$ and $B_n = b_n$ in (61). Then in (65). The second line cancels half of the integral.

Hence for this choice of the coefficients of f_N the square error, call it E^*

$$E^* = \int_{-\pi}^{\pi} f^2\, dx - \pi[2{a_0}^2 + \sum_{n=1}^{N}({a_n}^2 + {b_n}^2)] \tag{66}$$

Now substract (65) from (64) then the integral drop out and obtained the integral

$$E - E^* = \pi\{2(A_0 - a_o)^2 + \sum_{n=1}^{N}[(A_n - a_n)^2 + (A_n - a_n)^2]\} \tag{67}$$

Since the sum of square of real numbers on the right cannot be negative, so it follows that

$$E - E^* \geq 0 \Rightarrow E \geq E^* \text{ and } E = E^* \text{ if and only if } A_0 = a_o, A_n = a_n \text{ and } B_n = b_n, n = 1, 2, 3 \dots. N$$

From (66) E^* cannot increase as N increases, but may decrease. Hence with increasing N the partial approximationsto f, considered from the viewpoint of the square error.

Theorem 4.1: (mimimum square error): The square error of f_N in (61) with fixed N relative to f on the interval $-\pi \leq x \leq \pi$ is mimimum if and only if the coefficients of f_N in (60) are fourier cefficients of f. This minimum value is given by (66). Since $E^* \geq 0$ holds for every N, from (67), we obtained the important Bessel's inequality

$$2{a_0}^2 + \sum_{n=1}^{N}({a_n}^2 + {b_n}^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx \tag{68}$$

It can be shown that for such a function f, Parseval's theorem holds; that is, formula (64) holds with the equality sign, so that it becomes Parseval's identity

$$2{a_0}^2 + \sum_{n=1}^{N}({a_n}^2 + {b_n}^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx \tag{69}$$

Example 4.1: Approximate function $f(x)$ and compute the minimum square error E^* of $F(x)$ with $N = 1, 2, 3, \ldots, 100$ and 1000 and 10000 relative to

$$f(x) = x + \pi \quad -\pi \leq x \leq \pi.$$

Solution:

First compute the coefficients of fourier series and obtained as follows

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi + x)dx = \frac{1}{2\pi}\left(x\pi + \frac{x^2}{2}\right)\Big|_{-\pi}^{\pi} = \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi + x)\cos(nx)\,dx, n = 1, 2, \ldots$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\cos(nx)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}x\cos(nx)\,dx = \frac{1}{\pi}\left[\frac{\pi\sin(nx)}{n}\right]\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\left[\frac{x\sin(nx)}{n} - \int_{-\pi}^{\pi}\frac{x\sin(nx)}{n}dx\right]\Big|_{-\pi}^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{0}^{\pi}(x + \pi)\sin(nx)dx \text{ n=1, 2, 3,...}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\sin(nx)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}x\sin(nx)\,dx$$

$$\frac{1}{\pi}\left[\frac{\pi co(nx)}{n}\right]\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\left[\frac{x\sin(nx)}{n} - \int_{-\pi}^{\pi}\frac{x\sin(nx)}{n}dx\right]\Big|_{-\pi}^{\pi} = \frac{-2\cos(n\pi)}{n} = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = x + \pi \approx f_n = \pi + 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots + \frac{(-1)^{N+1}}{N}\sin Nx)$$

Thus $a_0 = \pi$, $a_n = 0$ and $b_n = \frac{2(-1)^{n+1}}{n} \Rightarrow {b_n}^2 = \frac{4}{n^2}$ approximation error can be determined by least square condition and by (67), we obtain that the following results.

$$E^*{}_n = \int_{-\pi}^{\pi}(x + \pi)^2\,dx - \pi\left[2\pi^2 + 4\sum_{n=1}^{N}\frac{1}{n^2}\right] = \frac{8\pi^3}{3} - \pi\left[2\pi^2 + 4\sum_{n=1}^{N}\frac{1}{n^2}\right]$$

$$E^*{}_1 = \frac{8\pi^3}{3} - \pi(2\pi^2 + 4) = 82.6834 - 74.5789 = 8.1045.$$

$$E^*{}_2 = 82.6834 - \pi\left(2\pi^2 + 4(1 + \frac{1}{2^2}\right) = 4.9629$$

$$E^*{}_3 = 82.6834 - \pi\left(2\pi^2 + 4(1 + \frac{1}{2^2} + \frac{1}{3^2}\right) = 3.5666$$

$$E^*{}_4 = 82.6834 - \pi\left(2\pi^2 + 4(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\right) = 2.7812 \quad E^*{}_5 = 82.6834 - \pi\left(2\pi^2 + 4(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}\right) = 2.2786$$

$$\vdots$$

$$E^*{}_N = \frac{8\pi^3}{3} - \pi\left[2\pi^2 + 4\sum_{n=1}^{N}\frac{1}{n^2}\right]$$

The Least square error at N^{th} partial sum is determined by MATLAB code given in appendix F yeild the following numerical result.

Table 3. Numerical values of least square error.

N	E^*_N	N	E^*_N	N	E^*_N	N	E^*_N	N	E^*_N	N	E^*_N
1	8.1045	6	1.9295	15	0.8105	40	0.3103	65	0.1918	90	0.1389
2	4.9629	7	1.6730	20	0.6129	45	0.2762	70	0.1782	95	0.1316
3	3.5666	8	1.4767	25	0.4927	50	0.2488	75	0.1664	100	0.1250
4	2.7812	9	1.3216	30	0.4120	55	0.2264	80	0.1561	1000	0.0126
5	2.2786	10	1.1959	35	0.3540	60	0.2077	85	0.1470	10000	0.0013

E^*_N = Least square error and N= order of partial sum.

MATLAB code presented in appendix D and E is used to plote the graph of approximation function and as shown in Figure 18.

Discusion:

Clearly from the Table 3 above E^*_N is decreasing with increasing N. Thus by including more terms of partial sum in fourier approximation that minimuze error resulting is best approximation of $f(x)$. Here, "best" means that the "error" of the approximation is as small as possible and it measure the goodness of agreement between approximation and actual function. The approximates f is quite well on the whole interval, except near $\pm\pi$ which $|f(x) - f_n(x)|$ is large at $\pm\pi$ and obey the Gibbs phenomenon.

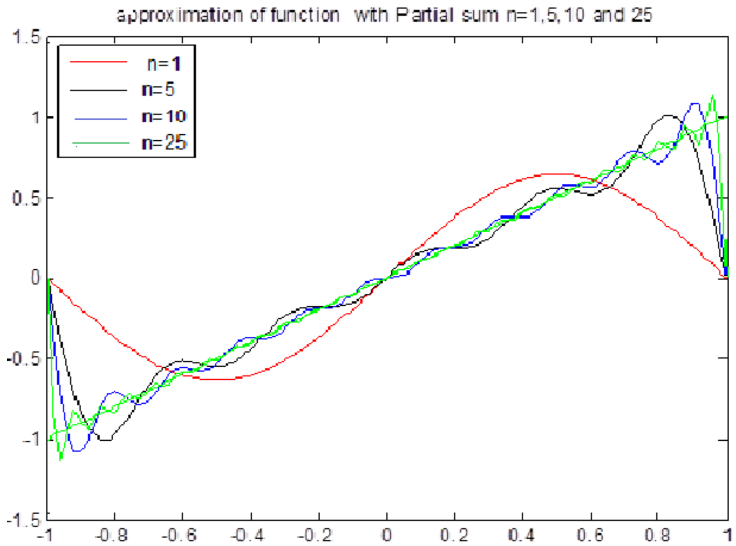

Figure 20. *Graph of approx function $f(x) = x + \pi$ with partial sum $n = 1, 5, 10$ and 25.*

4.5. Discrete Fourier Transformation

In using Fourier series, or trigonometric approximations we have to assume that a function $f(x)$ to be developed or transformed, is given on some interval, over which we integrate in the Euler formulas. Now very often a function $f(x)$ is given only in terms of values at finitely many points, and one is interested in extending Fourier series to discrete Fourier. The main application of such a "discrete Fourier analysis" concerns large amounts of equally spaced data, as they occur in telecommunication, time series analysis, and various simulation problems. There are many ways that the DFT arises in practice but generally one somehow arrives at a periodic sequence numbers. These numbers may arise, for example, extended periodically. They may also arise as a discrete set of values from the measurements in an experiment. Once again we would assume that they are extended periodically. In any case, the DFT of the sequence is a new periodic sequence and is related to the original sequence via a DFT inversion transform similar to the Inverse Fourier (DFT). Let $f(x)$ be period, of period 2π. We assume that N measurements of $f(x)$ are taken over the interval $0 \le x \le 2\pi$ at regularly spaced points

$$x_k = \frac{2\pi k}{N} \quad k = 0, 1, \dots, N-1 \tag{70}$$

At these points, we determine a complex trigonometric polynomial.

$$q(x) = \sum_{n=0}^{N-1} x(k) e^{inx_k} \tag{71}$$

That interpolate $F(x)$ at the node (69) that is $q(x_k) = F(x_k)$, denoting F_k by $F(x_k)$.

$$F_k = F(x_k) = q(x_k) = \sum_{n=0}^{N-1} f_n e^{inx_k} \quad k = 0, 1, \dots, N-1 \tag{72}$$

DFT produces a complex vector X of length N from an input vector x of length N by the following formula:

$$F_k = \sum_{n=0}^{N} f_n e^{inx_k} = X(k) = \sum_{n=0}^{N} x(n) e^{-i2\pi nk/N} \quad k = 0, 1, \dots, N-1$$

But in MATLAB cannot be use a zero or negative indices, so the sequences are $\{F_k\}_{k=1}^{N}$and the DFT is computed as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{\frac{-2\pi i}{N}(k-1)(n-1)} \text{ for } K = 1, 2, \dots, N \tag{73}$$

In MATLAB, the DFT is computed using the built-in function *fft*
In this case we have the formula for the Inverse Discrete Fourier Transform (IDFT) which gives

$$x(n) = \sum_{n=0}^{N-1} X(k)e^{\frac{2\pi i}{N}(k-1)(n-1)} \text{ for } n = 1, 2, \dots., N-1 \tag{74}$$

Example 4.2: Vector x = $[3, -2, 4, 5, 0, -1]$. Compute DFT and IDFT for N = 6.
Solution: MATLAB code in appendix G is used to calculate the DFT and IDFT and yield the following result.

Table 4. *Numerical values of DFT and IDFT.*

	k	vector	x	x(x)=DFT	IDFT
1.0000	3.0000	9.0000	3.0000		
2.0000	-2.0000	-5.5000	+	2.5981i	-2.0000
3.0000	4.0000	7.5000	-	4.3301i	4.0000
4.0000	5.0000	-	5.0000	5.0000	
5.0000	0.0000	7.5000	+	4.3301i	0.0000
6.0000	-1.0000	-5.5000	-	2.5981i	-1.0000

Relationship of the DFT to the Interpolation of a Data Set:
Let $x = [x_1, x_2, x_3, \dots., x_N]$ be a given vector of real values. Using these values from the following equally spaced data set on the interval [0, L]:

$t_1 = 0$	$t_{2=\Delta t}$	…..	$t_N = (N-1)\Delta t$	$t_{N+1} = \Delta t N = L$
x_1	x_2	…..	x_N	x_N

Theorem 4.2 (Interpolation Theorem)
The above data set is interpolated by the following trigonometric polynomial.

$$x(t) = a_0 + \sum_{k=1}^{k \le (N+1)/2} \left[a_k \cos\left(\frac{2\pi k}{L} t\right) + b_k \sin\left(\frac{2\pi k}{L} t\right)\right] \tag{75}$$

Where

$$a_0 = X(1)/N,\ a_k = \frac{2Real(X(k+1))}{N},\ b_k = \frac{-2Im(X(k+1))}{N} \text{ if N is even } a_{N/2} = \frac{X(N/2+1)}{N}$$

And

$$X = DFT(x) \tag{76}$$

Exampole 4.3:
Consider the vector x = $[1, -2, 3, 6, 0, -1]$ and Let L = 1, then the Data Set is given by:

0	1/6	1/3	½	2/3	5/6	1
1	-2	3	6	0	-1	1

The imlemanted MATLAB code shown in appendix H graphs of both this data set and the Trigonometric Polynomial defined in the Theorem. This MATLAB code displays the following data points and graph
Data =

0 0.1667 0.3333 0.5000 0.6667 0.8333 1.0000

1.0000 -2.0000 3.0000 6.0000 0 -1.0000 1.0000

X = DFT

7.0000 -8.0000 - 1.7321i 7.0000 + 3.4641i

1.0000 7.0000 - 3.4641i -8.0000 + 1.7321i

M =3

The coefficients are: $a_0 = 1.1667$, a_n = -2.6667 2.3333 0.1667, b_n = 0.5774 -1.1547 0

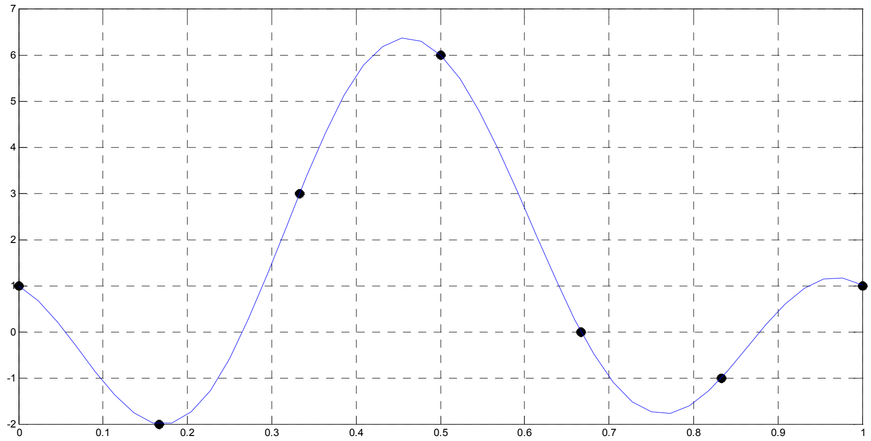

Figure 21. *Graph of approximation Fourier DFT.*

The Trigonometric Polynomial defined by these coefficient interpolates the date set. Note that the dot on the graph represented the date set and the graph passed througth the date points which represent the interpolate trigonometric polynomial.

5. Approximate Solution of PDES by FS Method

In this chapter we will study approximation solution of PDE by Fourier series method Fourier series were first developed to find solution PDE that arise in the study of physical system. Now we will consider the following PDE that involves a function $f(x_1, x_2, ... x_n)$ with no time dependent and the other functions $f(x_1, x_2, ... x_n, t)$ with time dependent. These equations are respectively known as Laplace's equation and the heat equation, these are the two most important partial differential equations, and much of the study of PDE's is devoted to understanding just these Two.

5.1. One Dimension Heat Equation: Solution by Fourier Series

Many heat conduction problems encountered in engineering applications involve time as an independent variable. The goal of analysis is to determine the variation of the temperature as a function of time and position u (x, t) within the heat conducting body. The temperature distribution in a medium depends on the conditions at the boundaries of the medium as well as the heat transfer mechanism inside the medium. To describe a heat transfer problem completely, two boundary conditions must be given for each direction of the coordinate system along which heat transfer is significant. Therefore, we need to specify two boundary conditions for one-dimensional problems, four boundary conditions for two dimensional problems.

In this section we study heat conducting bodies in one dimension heat equation and find temperature distribution in the body using FS method. We will to compare the solution obtains in FS with BCTS. Here first we investigate specifically solutions to selected special cases of the following form of the heat equation.

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \; 0 \leq x \leq L, t > 0 \tag{77}$$

Where $u = u(x, t)$, is the dependent variable, and c is the material property called thermal diffusivity which a constant coefficient. Equation (77) is a model of transient heat conduction in a slab of material with thickness L. The domain of the solution is a semi-infinite strip of width L that continues indefinitely in time t. In a practical computation, the solution is obtained only for a finite time, say t_{max}. Solution to Equation (77) requires specification of boundary at $x = 0$ and $x = L$, and initial conditions at $t = 0$. Simple boundary and initial conditions are

$$u(0, t) = u_0, u(L, t) = u_L, u(x, 0) = f(x) \tag{78}$$

The solution $u(x, t)$ of equation (77) satisfying conditions (78) is given in the three steps below.

Step1. Two ODES from heat equation (76)) substitution of a product $u(x, t) = F(x)G(t)$ into (77) gives

$$F\dot{G} = c^2 F'' \text{ G, With } \dot{G} = {}^{dG}/_{dt} \text{ and } F'' = d^2F/dx^2$$

To separate the variables, it can be dividing by c^2FG, we and obtained

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} \tag{79}$$

The left side depends only on t and the right side only on x, so that both sides must equal a constant k. we can see that for $k = 0$ or $k > 0$ the only solution $u = FG$ satisfying (78), is $u \equiv 0$. for negative $k = -p^2$ we obtain from (79),

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = -p^2.$$

Multiplication by the denominators immediately gives the two ODEs

$$F'' + p^2 F = 0 \tag{80}$$

And

$$\dot{G} + c^2 G = 0 \tag{81}$$

Step 2. Satisfying the boundary conditions (5.2). We first solve (5.4), general solution is

$$F(x) = A\cos px + B\sin px. \quad (82)$$

From the boundary conditions (78) it follows that $u(0,t) = F(0)G(t) = 0$ and $u(L,t) = F(L)G(t) = 0$. Since $G \equiv 0$ would give $u \equiv 0$, we require $F(0) = 0$, $F(L) = 0$ and get $F(0) = A = 0$. by (78) and then $F(L) = B\sin pL = 0$ and $B \neq 0$ (to avoid $f \equiv 0$). Thus $sinp\ L = 0$ Hence $p = \frac{n\pi}{L}$ $n = 1, 2, \ldots$

Setting $B = 1$, we obtain the following solutions of (80) satisfying (78)

$$F_n(x) = \sin\frac{n\pi x}{L} \quad n = 1, 2, \ldots$$

Now solve (80) for $p = {n\pi}/{L}$, we get

$$\dot{G} + \lambda^2{}_n G = 0 \text{ where } \lambda_n = \frac{cn\pi}{L}.$$

General solution is

$$G(t) = B_n e^{-\lambda_n{}^2 t} \quad n = 1, 2, \ldots.$$

Where B_n is a constant. Hence the functions

$$u_n(x,t) = F_n(x)G_n(t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n{}^2 t} \quad n = 1, 2, \ldots \quad (83)$$

The solution of heat equation (77) Satisfying conditions (77) These are the eiginfunction of the problem, corresponding to the eigenvalue $\lambda_n = {cn\pi}/{L}$.

Step 3. Solution of the entire problem. Fourier series. So far we have solution (5.7) satisfying the boundary conditions. To obtain a solution that also satisfies the initial condition. We consider a series of these eigenfunctions.

$$u(x,t) = \sum_n^\infty u_n(x,t) = \sum_{n=1}^\infty B_n \sin\frac{n\pi x}{L} e^{-\lambda_n{}^2 t} \ (\lambda_n = {cn\pi}/{L}) \quad (84)$$

From (78), we have

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x).$$

Here for (84) to satisfy (78) the B_n's must be the coefficient of the Fourier series and given by

$$B_n = \frac{2}{L}\int_0^L f(x)\ \sin\frac{n\pi x}{L} dx \quad n = 1, 2, \ldots \quad (85)$$

Example 5.1:

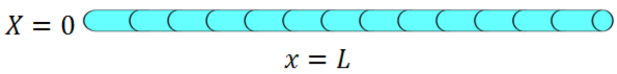

Figure 22. *A rod of length L whose ends are kept at 0°C.*

Find the temperature in a insulated rod of length L whose ends are kept at temperature 0. Assume that there is no heat source or sink in the rod and the initial temperature is

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{L}{2} \\ L - x & \frac{L}{2} \leq x \leq L \end{cases}.$$

The end of the rod are then connected to insulators to maintain the ends at $u(0,t) = 0$ and $u(L,t) = 0$. and plot temperature distribution at $t = 0, 0.2, 0.5, 1$ and 9.

Solution: From (85) we get

$$B_n = \frac{2}{L}\int_0^L f(x)\ \sin\frac{n\pi x}{L} dx \quad n = 1, 2, \ldots$$

$$= \frac{2}{L}\int_0^{L/2} x\ \sin\frac{n\pi x}{L} dx + \frac{2}{L}\int_{L/2}^{L} (L-x)\ \sin\frac{n\pi x}{L} dx$$

$$= \frac{2}{L}\left(-\frac{L}{n\pi}\right)\left(x\cos\frac{n\pi x}{L}\Big|_0^{\frac{L}{2}} - \int_0^{\frac{L}{2}} \cos\frac{n\pi x}{L} dx + (L-x)\cos\frac{n\pi x}{L}\Big|_{\frac{L}{2}}^{L} + \int_{L/2}^{L} \cos\frac{n\pi x}{L} dx\right)$$

$$= -\frac{2}{n\pi}\left(\int_{L/2}^{L} \cos\frac{n\pi x}{L} dx - \int_0^{L/2} \cos\frac{n\pi x}{L} dx\right) = \frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$\text{Thus } B_n = \begin{cases} \frac{4L}{n^2\pi^2}, n - 1,5,9, \ldots \\ -\frac{4L}{n^2\pi^2} \ n = 3, 7, 11, \ldots \\ 0 \ n = 2, 4, 6, \ldots. \end{cases}$$

Hence from heat (84) fourier solution is

$$u(x,t) = u(x,t) \approx \sum_n^N u_n(x,t) = \sum_{n=odd}^N \frac{4}{n^2\pi^2} \sin\frac{n\pi x}{3} e^{-\lambda^2{}_n t}$$

$$= \sum_{n=1,5,9}^N \frac{4L}{n^2\pi^2} \sin\frac{n\pi x}{L} e^{-\lambda^2{}_n t} - \sum_{n=3,7.11}^N \frac{4L}{n^2\pi^2} \sin\frac{n\pi x}{L} e^{-\lambda^2{}_n t}$$

Where

$$(\lambda_n = {cn\pi}/{L}) \ n = 1, 2, \ldots \text{ and } c = 1$$

MATLAB code to plot the graph of solution is given in appendix I and Figure 23 shows the graph of the solution of given problem for $t = 0, 0.2, 0.5, 1$ and 9.

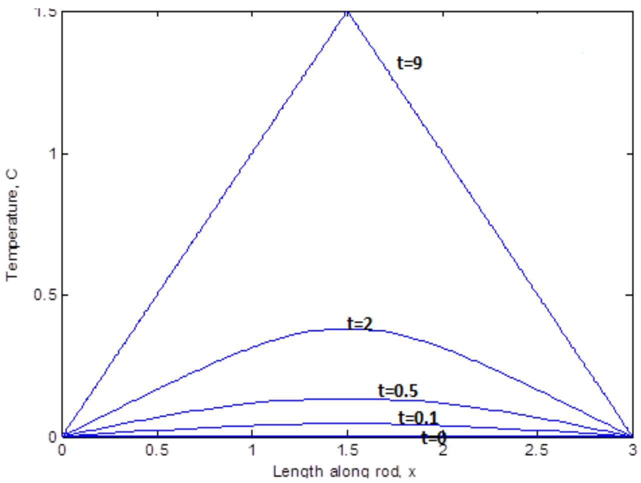

Figure 23. *Temperature distributions of 1D heat condition equations.*

5.2. Bar with Two Ends Kept at Arbitrary Temperatures: Non-homogeneous BC

Here we will discuss the heat conduction with initial boundary value problems with non homogeneous boundary conditions. Let us consider example 5.1 where the bar are ends are kept at arbitrary constant temperatures of T_1 degrees at the left end, and T_2 degrees at the right end. The heat conduction problem is therefore given by the initial boundary value problem:

$$u_t = c^2 u_{xx},\ 0 < x < L, t > 0,$$

$$u(0,t) = T_1 \text{ And } u(L,t) = T_2 \tag{86}$$

$$u(x,0) = f(x)$$

The boundary conditions are now non-homogeneous unless T_1 and T_2 are both 0, at least one of the boundary values are nonzero. The non-homogeneous boundary conditions are rather easy to work with, more so than we might have reasonably expected. First, let us be introduced to the concept of the steady state solution. It is the part of the solution $u(x,t)$ that is independent of the time variable t. Therefore, it is a function of the spatial variable alone. The solution $u(x,t)$ of the given problem is the sum of two parts, a time independent part and a time dependent part:

$$u(x,t) = v(x) + w(x,t). \tag{87}$$

Where, $v(x)$ is the steady state solution, which is independent of t, and $w(x,t)$ is called the transient solution, which does vary with t.

5.3. The Steady-State Solution

The steady state solution, $v(x)$, of a heat conduction problem is the part of the temperature distribution function that is independent of time t. It represents the equilibrium temperature distribution. $v(x)$ is a function of x alone and satisfy the heat conduction equation. Since $v_{xx} = v''$ and $v_t = 0$, substituting them into the heat conduction equation, we get

$$c^2 v_{xx} = 0 \tag{88}$$

Divide both sides by c^2 and integrate twice with respect to x, we find that $v(x)$ must be in the form of a degree 1 polynomial:

$$v(x) = Ax + B \tag{89}$$

Then, rewrite the boundary conditions in terms of v: $u(0,t) = v(0) = T_1$, and $u(L,t) = v(L) = T_2$. Apply those two conditions to find that:

$$v(0) = T_1 = A(0) + B = B \rightarrow B = T_1$$

$$v(L) = T_2 = AL + B = AL + T_1 \rightarrow A = (T2 - T1)/L$$

Therefore,

$$v(x) = \frac{T_2 - T_1}{L} x + T_1 \tag{90}$$

One can clearly understand from (90) the steady state solution is a time independent function. It is obtained by setting the partial derivative(s) with respect to t in the heat equation constant zero, and then solving the equation for a function that depends only on the spatial variable x. The term steady implies no change with time at any point within the medium, while transient implies variation with time or time dependence. Since we have already found $v(x)$, the change in the boundary conditions (BC):

$$u(0,t) = T_1 = v(0) + w(0,t) \rightarrow w(0,t) = T_1 - v(0) = 0$$

$$u(L,t) = T_2 = v(L) + w(L,t) \rightarrow w(L,t) = T_2 - v(L) = 0$$

Note: Recall that $u(0,t) = v(0) = T_1$, and $u(L,t) = v(L) = T_2$.
Change in the initial condition (IC):

$$u(x,0) = f(x) = v(x) + w(x,0) \rightarrow w(x,0) = f(x) - v(x) \tag{91}$$

Consequently, the transient solution is a function of both x and t that must satisfy the new initial boundary value problem:

$$c^2 w_{xx} = w_t\ 0 < x < L, t > 0,$$

$$w(0,t) = 0, \text{ and } w(L,t) = 0 \tag{92}$$

$$w(x,0) = f(x) - v(x)$$

We notice that the new problem just described is precisely the same initial boundary value problem associated with the heat conduction of a bar with both ends kept at 0 degree. Therefore, the transient solution $w(x,t)$ of the current problem is just the general solution of the previous heat conduction problem with homogeneous boundary conditions that of a bar with two ends kept constantly at 0 degree:

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda^2 n^2 \pi^2 t/L^2} \quad (93)$$

Where the coefficients C_n are equal to the corresponding Fourier sine coefficients b_n of the newly rewritten initial condition $w(x,0) = f(x) - v(x)$. Thus $w(x,0)$ is odd periodic extension, of period 2L. Explicitly, they are given by

$$c_n = b_n = \frac{2}{L}\int_0^L (f(x) - v(x)) \sin\left(\frac{n\pi x}{L}\right) dx \;\; n = 1, 2, 3, \ldots \quad (94)$$

Finally, combining the steady state and transient solutions together, the general solution of the temperature distribution of a bar whose ends are kept at T_1 degrees at the left, and T_2 degrees at the right, becomes

$$u(x,t) = v(x) + w(x,t) = \frac{T_2 - T_1}{L} x + T_1 + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda^2 n^2 \pi^2 t/L^2} \quad (95)$$

Example 5.2: The temperatures at the end $x = 0$ and $x = L$ of a 100cm long rod with insulated sides are held at temperatures of 0 and 100, respectively until reaching steady state. Then the temperature at the ends is interchanged. Find $T(x,t)$.

Solution:

The solution to the problem of $T = T_1$ at $x = 0$ and $T = T_2$ at $x = 100, is$

$$T(x,t) = T_1 + (T_2 - T_1)\frac{x}{l} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l} x\right) e^{-\frac{n^2\pi^2}{l^2} t}$$

Where

$$b_n = \frac{2}{L}\int_0^L (f(x) - v(x)) \sin\left(\frac{n\pi x}{L}\right) dx$$

This can be applied directly note that here $f(x) = 100x/L$ is the initial condition obtained from the previous steady state and that $T_1 = 100$ and $T_2 = 0$. The Fourier coefficients are then

$$b_n = \frac{1}{50}\left[-\frac{100^2}{n\pi}(-1)^n\right] - \frac{200}{n\pi} = \frac{200}{n\pi}[(-1)^{n+1} - 1] = \begin{cases} -\frac{400}{n\pi} \; if\; n\; is\; even \\ 0 \; if\; n\; is\; odd \end{cases}$$

The result is

$$T(\mathrm{x},\mathrm{t}) = 100 - \mathrm{x} - \sum_{\mathrm{n}>1,\mathrm{even}}^{\infty} \frac{400}{\mathrm{n}\pi} \sin\left(\frac{\mathrm{n}\pi}{100}\mathrm{x}\right) \mathrm{e}^{-\frac{\mathrm{n}^2\pi^2\sigma}{100^2}\mathrm{t}} = 100 - \mathrm{x} - \sum_{\mathrm{m}=1}^{\infty} \frac{400}{2\mathrm{m}\pi} \sin\left(\frac{2\mathrm{m}\pi}{100}\mathrm{x}\right) \mathrm{e}^{-\frac{(2\mathrm{m})^2\pi^2\sigma}{100^2}\mathrm{t}}$$

$$= 100 - \mathrm{x} - \frac{200}{\pi} \sum_{\mathrm{m}=1}^{\infty} \frac{1}{\mathrm{m}} \sin\left(\frac{\mathrm{m}\pi}{50}\mathrm{x}\right) \mathrm{e}^{-\frac{\mathrm{m}^2\pi^2\sigma}{2500}\mathrm{t}}$$

5.4. Implicit Backward Euler Method and FS for 1D Heat Equation

In this section we will study the numerical solutions to the heat equation using implicit scheme that is backward time centered space (BTCS) method, and applied to a simple problem involving the one-dimensional heat equation. We will compare the result obtained from Fourier series method with BTCS method for one dimension heat equation.

Consider the 1 D heat equation.

$$u_t = cu_{xx} \quad (96)$$

We can solve this PDE for points on a grid using the finite difference method where we discretise in x and t of $0 \le x \le a$ and $0 \le t \le T$

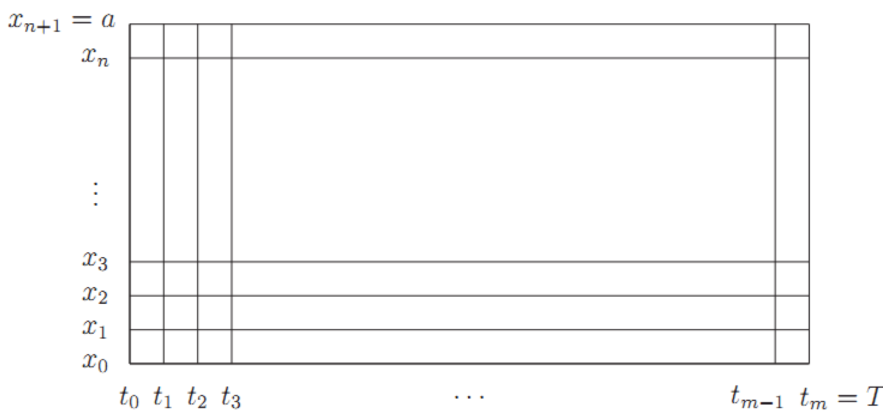

Figure 24. *Discretise in time step and grid spacing.*

We discretise in time with time step $\Delta t = {}^{T}/_{m}$ and in space with grid spacing

$\Delta x = {}^{a}/_{n+1}$. And let $t_k = k\Delta t$ where $0 \leq k \leq$ m and $x_j = j\Delta x$ where $0 \leq j \leq$ n+1.

Let $u^{k+1} = u(t_{k+1}, x_j)$ then the finite difference approximations for equation (97) are given by:

$$u_{xx}(t_{k+1}, x_j) = \frac{u_{j-1}{}^{k+1} - 2u_j{}^{k+1} + u_{j+1}{}^{k+1}}{\Delta x^2} \tag{97}$$

$$u_t(t_{k+1}, x_j) = \frac{u_j{}^{k+1} - u_j{}^{k}}{\Delta x^2} \tag{98}$$

Equation (96) become

$$u_j{}^k = u_j{}^{k+1} - \frac{c\Delta t}{\Delta x^2}\left[u_{j-1}{}^{k+1} - 2u_j{}^{k+1} + u_{j+1}{}^{k+1}\right] = (1+2r)u_j{}^{k+1} - r(u_{j-1}{}^{k+1} + u_{j+1}{}^{k+1}) \tag{99}$$

Where $r = \frac{c\Delta t}{\Delta x^2}$

We still need to solve for $u_j{}^{k+1}$ given $u_j{}^k$ is known. Implies that this requires solving tridiagonal linear system of n equations.

Again we let $u_j{}^k = u(x_j, t_k)$: $x_j = j\Delta x, j = 0, 1, 2, \ldots, n+1, \Delta x = \frac{a}{n+1}$

$$t_k = k\Delta t, k = 0, 1, 2, \ldots, m, \Delta t = \frac{T}{m}.$$

Boundary conditions (Dirichlet)

$$u_0{}^k = 0, u_{j+1}{}^k = 0$$

System of equations

$$-r^{k+1} + (1+2r)u_j{}^{k+1} - ru_{j-1}{}^{k+1} = u_j{}^k, \text{j} \in 1, 2, \ldots., m, k \in 0, 1, \ldots, N.$$

$$u_0{}^k = 0, u_{j+1}{}^k = 0 \; k \in 0, 1, \ldots, N$$

$$Au^{k+1} = u^k \tag{100}$$

Where A is represented tridiagonal matrix.

Example 5.3: Consider the one-dimensional heat equation, $u_t = u_{xx}, 0 < x < 1, t > 0$ subject to homogeneous Dirichlet boundary conditions:

$$u(0,t) = 0, u(1,t) = 0$$

And the initial condition:

$$u(x,0) = f(x) = \begin{cases} 2x \; 0 < x \leq 1/2 \\ 2 - 2x \; 1/2 \leq x \leq 1 \end{cases}$$

Approximate the solution $u(x,t)$ using the implicit finite difference scheme consisting of a backward difference in time and centered difference in space.

Solution:

Suppose we choose $N = 4$ intervals on [0, 1] and set $x_0 = 0$, $x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}$ and $x_4 = 1$. Let $u_j{}^m$ denote an approximation to the exact solution $u(x_j, t_m)$. If we set $t_0 = 0$ then the implicit finite difference schemes based on centered differences in spaces and backward difference in time yield equation (97) for approximations to $u(x,t)$ at the interior space nodes, at each new interval level t_m. We have:

$$u_j{}^m = -r^{m+1} + (1+2r)u_j{}^{m+1} - ru_{j-1}{}^{m+1} = u_j{}^m, \text{j= } 1{:}3, m = 1, 2, \ldots$$

where $r = \frac{k}{h^2}$. The boundary conditions gives values for the end points at each time level. $u_0{}^m = u_4{}^m = 0, m = 1, 2, \ldots.$ with $h = \frac{1}{4}$, we obtain three equations for unknown values $u_1{}^{m+1}, u_2{}^{m+1}, u_3{}^{m+1}$ at each new time step:

$$(1+32k)u_1{}^{m+1} - 16ku_2{}^{m+1} = u_1{}^m - 16ku_1{}^{m+1} + (1+32k)u_2{}^{m+1} - 16ku_3{}^{m+1}$$

$$= u_2{}^m - 16ku_2{}^{m+1} + (1+32k)u_3{}^{m+1} = u_3{}^m$$

$$\Rightarrow \begin{pmatrix} 1+32k & -16k & 0 \\ -16k & 1+32k & -16k \\ 0 & -16k & 1+32k \end{pmatrix} \begin{pmatrix} u_1{}^{m+1} \\ u_2{}^{m+1} \\ u_3{}^{m+1} \end{pmatrix} = \begin{pmatrix} u_1{}^m \\ u_2{}^m \\ u_3{}^m \end{pmatrix}$$

If we set $t_0 = 0$ and choose $k = 0.01$ and notice that the initial condition gives:

$$u_1{}^0 = f(x_1) = 1/2, u_2{}^0 = f(x_2) = 1, u_3{}^0 = f(x_3) = 1/2.$$

$$\begin{pmatrix} 1.32 & -0.16 & 0 \\ -0.16 & 1.32 & -0.16 \\ 0 & -16k & 1.32 \end{pmatrix}\begin{pmatrix} u_1{}^1 \\ u_2{}^1 \\ u_3{}^1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

We then have to solve the 3 × 3 linear systems

$$\begin{pmatrix} 1.32 & -0.16 & 0 \\ -0.16 & 1.32 & -0.16 \\ 0 & -16k & 1.32 \end{pmatrix}\begin{pmatrix} u_1{}^1 \\ u_2{}^1 \\ u_3{}^1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

Using Gaussian elimination method, we have $u_1{}^1 = 0.489, u_2{}^1 = 0.8751, u_3{}^1 = 0.4849$.

Example 5.4: Heat generated from an electric wire is defined by the time-dependent heat equation:

$u_t = \lambda^2 u_{xx}$ With $u(0,t) = u(L,t) = 0$ and $u(x,0) = 0$.

Use Fourier series method and the implicit scheme with $N = 10$, $L = 2, \lambda = 1$ and $p = 1$, and find the temperature distribution in the electric wire and plot the temperature distribution at t = 0, 0.02, 0.1, 0.2, 2. Take $dt = 0.001$ and compare the result obtained in both methods.

Solution:

i) Fourier series method

Solution for this problem through Fourier series approximation:

Consider the BC

$$= \lambda^2 u_{xx} + p, \text{where p is constant.} \tag{101}$$

$$u(x,0) = 0, u(0,t) = 0, u(L,t) = 0$$

We assume a solution in the form

$$u(x,t) = u(x,t) + U(x,t)$$

We substitutition in to the equation (101), yield

$$u_t = \lambda^2(u_{xx} + U_{xx}) + p$$

And if $U\ (x)$ satisfies the equation

$$\lambda^2 U_{xx} + p = 0$$

Then $u(x,t)$ satisfies the heat equation

$$u_t = \lambda^2 u_{xx} \tag{102}$$

Then, the solution of the system with BC

$$\lambda^2 U_{xx} + p = 0, U\ (0) =\ 0, U\ (L) = 0,$$

By integrating two times, we obtain

$$U(x) = \frac{p(xL - x^2)}{2\lambda^2} = \frac{pxL}{2\lambda^2} - \frac{x^2 p}{2\lambda^2} \tag{103}$$

Now the solution in (102) is given by

$$v(x,t) = \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{L} e^{\frac{-n^2\pi^2 t}{L^2}t} \tag{104}$$

and with coefficient

$B_n = \frac{2}{L}\int_0^L \left[\frac{p(xL-x^2)}{2\lambda^2}\right] \sin\left(\frac{n\pi x}{L}\right) dx$, integration by part yield

$$= \begin{cases} \frac{4L^2 p}{n^3\pi^3 c^2} \text{ for n odd} \\ 0 \text{ for n even} \end{cases} \tag{105}$$

Thus from (`103) and (104) the solution in FS method is given by

$$u(x,t) \approx -\frac{px^2}{2\lambda^2} + \frac{pLx}{2\lambda^2} + \sum_{n=odd}^{N} \frac{4L^2 p}{n^3\pi^3} sin\frac{n\pi x}{L} e^{-\frac{n^2\pi^2 t}{L^2}} \tag{106}$$

ii) Implicit scheme

The fully implicit distribution scheme is

$$\lambda^2 \frac{u_{i-1}{}^{n+1} - 2u_i{}^{n+1} + u_{i+1}{}^{n+1}}{\Delta x^2} + 1 = \frac{u_i{}^{n+1} - u_i{}^n}{\Delta t} \tag{107}$$

Equation (107) is called the Back ward Time, Centered Space or BTCS approximation to the heat equation. A slight improvement in computational efficiency can be obtained with a small rearrangement of Equation (107)

$$-ru_{j+1}{}^{k+1} + (1+2r)u_j{}^{k+1} - ru_{j-1}{}^{k+1} + 1 = u_j{}^k$$

$$\text{j} \in 1,2,\ldots,m, k \in 0,1,\ldots,N. \tag{108}$$

$$u_0{}^k = 0, u_{j+1}{}^k = 0\ k \in 0,1,\ldots,N$$

Where $r = \frac{\lambda\Delta t}{\Delta x^2}$

The BTCS scheme is easy to implement because the values of $u_j{}^{k+1}$ can be updated independently of each other. The entire solution is contained in two loops: an outer loop over all time steps and an inner loop over all interior Gerald (2011). The code in appendix J shows how easy it is to implement the BTCS scheme.

Finally, we note that Equation (108) can be expressed as a matrix multiplication

$$Au^{k+1} = u^k$$

$$\Rightarrow u^{k+1} = A^{-1}u^k$$

Where A^{-1} is tridiagonal matrix and

Implementation of the BTCS scheme requires solving a system of equations at each time step and MATLAB code seen in Appendix J is used to approximate the temperature and plot temperature distribution in wire. As the result numerical solution in both method given in Table 5 and presented in graph as seen in Figure 25 below.

Graphical representation

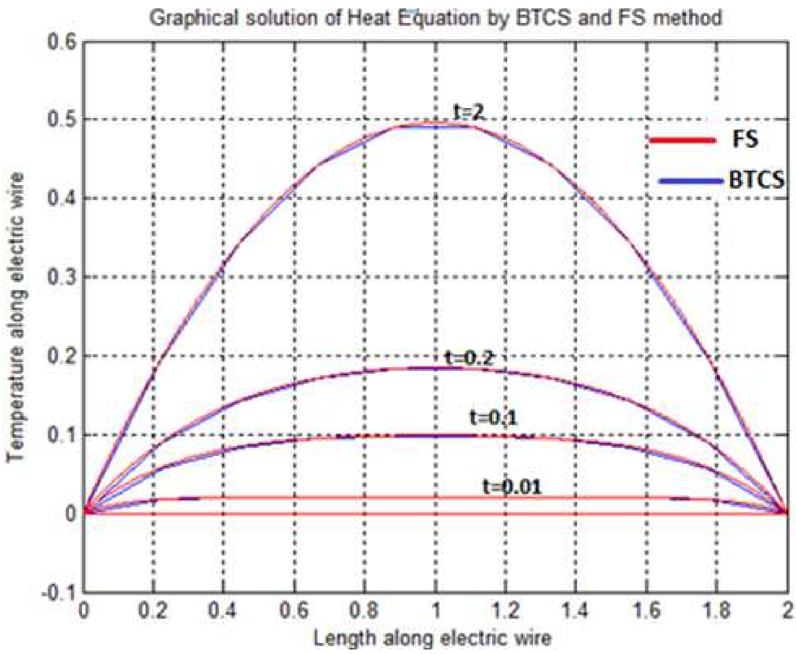

Figure 25. *Comparison of FS with BTCS of solution of temperature distribution in electric wire.*

Table 5. *Numerical solution of temperature distribution in electric wire.*

X	t=0		t=0.02		t=0.1		t=0.2		t=2	
	FS	BTCS	FS	BTCS	FS	BTCS	FS	BTCS	FS	BTCS
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.2	0.0282	0.0290	0.0554	0.0493	0.0826	0.0786	0.0825	0.0786	0.1789	0.1784
0.4	0.0313	0.0411	0.0830	0.0749	0.1348	0.1278	0.1346	0.1278	0.3178	0.3170
0.6	0.0226	0.0462	0.0938	0.0875	0.1651	0.1566	0.1651	0.1566	0.4170	0.4159
0.8	0.0129	0.0482	0.0965	0.0931	0.1804	0.1714	0.1805	0.1714	0.4765	0.4752
1	0.0088	0.0487	0.0968	0.0947	0.1850	0.1759	0.1852	0.1759	0.4963	0.4949
1.2	0.0129	0.0482	0.0965	0.0931	0.1651	0.1714	0.1805	0.1714	0.4765	0.4752
1.4	0.0226	0.0462	0.0938	0.0875	0.1651	0.1566	0.1651	0.1566	0.4170	0.4159
1.6	0.0313	0.0411	0.0830	0.0749	0.1348	0.1278	0.1346	0.1278	0.3178	0.3170
1.8	0.0282	0.0290	0.0554	0.0493	0.0826	0.0786	0.0825	0.0786	0.1789	0.1784
2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

t=Time, FS=Fourier series and BTCS=Back ward Time Centered Space.

Discussion:

We obtain the solution of heat equation using Fourier series and BTCS method and the results we obtain are presented in the Table 5 and Figure 25, As we compare the results obtain in both methods, clearly we see that approximation solutions are almost the same, but small difference is observed, this is due to the truncation error occurred in implicit scheme. Table 5 and Figure 25 also show that temperature in electric wire is not uniformly distributed and it is varying with time and high temperature in wire is observed at $t = 2$ but low temperature in wire is seen at $t = 0$. From the figure 25 we see that the graph of the solution of FS is smooth curve as compared to the graph of the solution of BTCS, which shows the solution of FS method is equal to an analytical solution and it agrees with actual solution. Therefore, we can conclude that FS method is more accurate than numerical solution of BTCS method.

5.5. Laplace Equation

We consider the two-dimensional heat equation given by

$$\frac{\partial u}{\partial t} = c^2\, \Delta^2\, u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{109}$$

For steady (that is time-independent) problem. $\frac{\partial u}{\partial t} = 0$ and the heat equation reduce to Laplace's equation

$$\frac{\partial u}{\partial t} = \Delta^2\, u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{110}$$

The heat problem is considered in some region R of the xy-plane and a given boundary condition on the boundary curve C of R.

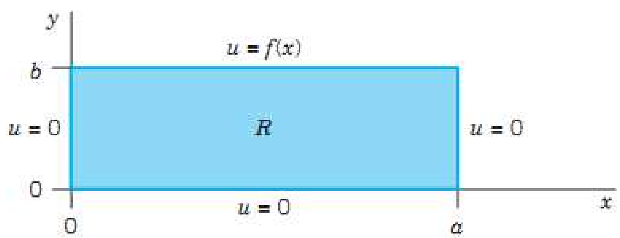

Figure 26. *Rectangle R and given boundary values.*

We consider a Dirichlet problem for Laplace's equation (5110) in a rectangle R, assuming that the temperature $u(x, y)$ equals given functions on the upper side and 0 on the other three sides of the rectangle. We solve this problem by separating variables. Substituting $u(x, y) = F(x)G(y)$ into (110) written as $u_{xx} = -u_{yy}$ dividing by FG, and equating both sides to a negative constant, we obtain

$$\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2} = -K \qquad (111)$$

From this we get

$$\frac{d^2F}{dx^2} + KF = 0. \qquad (112)$$

And the left and right boundary conditions imply

$$F(0) = 0. \text{ and } F(a) = 0 \qquad (113)$$

This give $k = (n\pi/a)^2$ and corresponding nonzero solutions

$$F(x) = F_n(x) = \sin\frac{n\pi}{a}x \; n = 1, 2, \ldots \qquad (114)$$

The ODE for G with $k = (n\pi/a)^2$ then becomes

$$\frac{d^2G}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G = 0$$

Solutions are

$$G(y) = G_n(y) = A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}$$

Now the boundary condition $u = 0$ on the lower side of R implies that $G_n(0) = 0$ that is $G_n(0) = A_n + B_n = 0$ or $A_n = -B_n$. This gives

$$G_n(y) = A_n(e^{n\pi y/a} - e^{n\pi y/a}) = 2A_n \sin h\frac{n\pi y}{a}.$$

From this and (113) $2A_n = A^*_n$ we obtain as the eigenfunction of our problem

$$u_n(x, y) = A^*_n \sin\frac{n\pi x}{a} \sin h\frac{n\pi y}{a}. \qquad (115)$$

These solutions satisfy the boundary condition $u = 0$ on the left, right, and lower sides.

To get a solution satisfying the boundary condition $u(x, b) = f(x)$ on the upper side, we consider the infinite series $u(x, y) = \sum_{n=1}^{\infty} u_n(x, y)$.

From this and (115) with $y = b$ we obtain $u(x, b) = \sum_{n=1}^{\infty} A^*_n \sin\frac{n\pi x}{a} \sin h\frac{n\pi b}{a}$

We can write this in the form $u(x, b) = \sum_{n=1}^{\infty}\left(A^*_n \sin h\frac{n\pi b}{a}\right) \sin\frac{n\pi x}{a}$

This shows that the expressions in the parentheses must be the Fourier coefficients b_n of $f(x)$ and by Euler formula

$$b_n = \int_0^a f(x)\sin\frac{n\pi x}{a}dx = A^*_n \sin h\frac{n\pi b}{a}$$

From this and (115) we see that the solution of our problem is

$$u(x, y) = \sum_{n=1}^{\infty} A^*_n \sin\frac{n\pi x}{a} \sin h\frac{n\pi b}{a} \qquad (116)$$

where

$$A^*_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx. \qquad (117)$$

Example 5.5: A rectangular steel plate is bounded by $x = 0, x = 1, y = 0, y = 1$ with the following boundary condition: $u(x, 0) = 0$, $u(1, y) = 0$, $u(0, y) = 0$ and $u(x, 1) = 200$. If one of edges is held at 200°C and the other three edges are held at 0°C. What are the steady state temperature at interior points and plot the temperature distribution in rectangular region R.

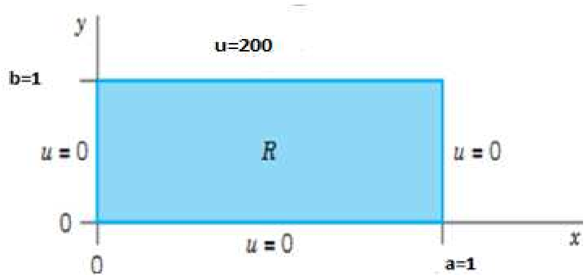

Figure 27. *Rectangle R with boundary condition.*

Solution:

The steady-state temperature distribution is governed by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Numerical solution of temperature distribution through the rectangle R is given by (116)

Since from boundary candition $a = b = 1$. We get

$$u(x, y) \approx \sum_{n=1}^{N} A^*_n \sin(n\pi x)\sin h(n\pi y)$$

where

$$A^*_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx = \frac{2}{\sinh(n\pi)}\int_0^1 200 \sin n\pi x \, dx.$$

$$= \frac{400}{\sinh(n\pi)}\left(\frac{-\cos n\pi}{n\pi} + \frac{\cos 0}{n\pi}\right) = \begin{cases} \frac{800}{n\pi\sinh(n\pi)} & if\ n\ is\ odd \\ 0\ if\ n\ is\ even \end{cases}$$

$$\Rightarrow A^*_n = \frac{800}{n\pi\sinh(n\pi)}.$$

$$u(x, y) \approx \sum_{n=1}^{N} \frac{800}{n\pi\sinh(n\pi)} \sin(n\pi x)\sin h(n\pi y).$$

Numerical solution is obtaind by using MATLAB code given in appendix K.

Table 6. *Numerical result of heat equation by FS methods.*

x\y	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	
0	0	0	0	0	0	0	0	0	0	0	0	
0.1	0	2.1881	4.1532	5.7014	6.6883	7.0268	6.6883	5.7014	4.1532	2.1881	0	
0	.2	0	4.6044	8.7318	11.9740	14.0346	14.7402	14.0346	11.9740	8.7318	4.6044	0
0.3	0	7.5099	14.2152	19.4493	22.7553	23.8831	22.7553	19.4493	14.2152	7.5099	0	
0.4	0	11.2452	21.2071	28.888	33.6823	35.3063	33.6823	28.89	21.2071	11.2452	0	
0.5	0	16.3177	30.5507	41.2682	47.8119	50.0000	47.8119	41.2682	30.5507	16.318	0	
0.6	0	23.6035	43.5513	57.9007	66.3177	69.0699	66.3177	57.9007	43.5513	23.6035	0	
0.7	0	34.9042	62.4404	80.5507	90.4560	93.5804	90.4560	80.5507	62.4404	34.904	0	
0.8	0	54.790	91.2682	111.3704	121.2071	124.1584	121.2071	111.370	91.2682	54.79	0	
0.9	0	97.8119	136.4525	151.8845	158.4631	160.3379	158.4631	151.884	136.4525	97.81	0	
1	0	196.683	199.35	200.5075	201.108	201.22	201.108	200.507	199.346	196.682	0	

Graphical representation

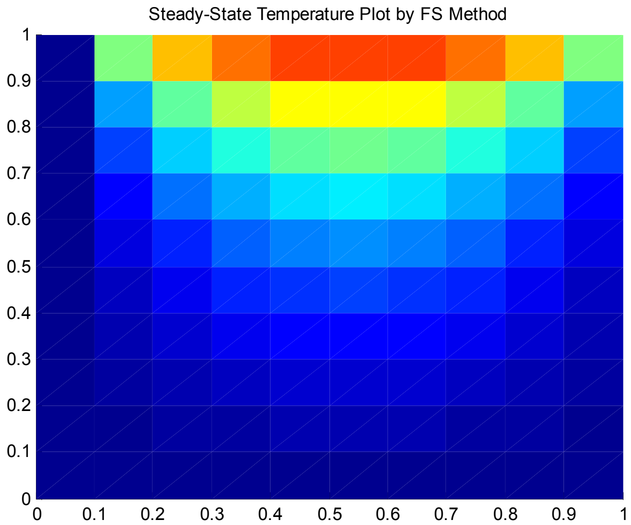

Figure 28. *Temperature distribution in rectangle R by FS method.*

5.6. Fourier Series and Iterative Method for Laplace Equation

The most commonly used approximation methods for the solution of two dimensional heat equations is iterative method. Since we have two systems that depend on each other, we need some technique of iteration between the solving of the first and the second equation. Jacobi iterative is a method that is used to find the numerical solution of heat equation. Thus, we will compare the results obtain from Fourier series method and iterative method for Laplace equation. The steady –state heat flow equation in two dimensions is given by,

$$\Delta^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Using the difference method the above equation can be written as

$$\Delta^2 u = \frac{1}{h^2}\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\} u_{ij} = 0 \qquad (118)$$

At certain set of grid points (x_i, y_j) gives the set of simultaneous linear equation. Proper ordering of the equation gives a diagonally dominant system. And with a small rearrangement of equation (118), we get

$$\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\} u_{ij} = 0 \rightarrow u_{ij} = \frac{u_{i+j,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}}{4} \quad (119)$$

Equation (118) is called five point formulas. The system of algebraic equations is readily solved using iterative methods. In the Jacobi method for the numerical solution of Laplace's equation in a uniform mesh one assumes a solution for the interior nodes and then computes an improved approximation using the five point formula, i.e.

$${u_{ij}}^{k+1} = \frac{{u^k}_{i+1,j}+{u^k}_{i-1,j}+{u^k}_{i,j+1}+{u^k}_{i,j-1}}{4} \qquad (120)$$

To find the solution for a two-dimensional Laplace equation simply:

Initialize u_{ij} to some initial guess.

Apply the boundary conditions.

For each internal mesh point set

$${u_{ij}}^{k+1} = \frac{{u^k}_{i+1,j}+{u^k}_{i-1,j}+{u^k}_{i,j+1}+{u^k}_{i,j-1}}{4}$$

Replace old solution ${u_{ij}}^k$ with new estimate ${u_{ij}}^{k+1}$.

If solution does not satisfy tolerance, repeat from step ii.

Example 5.6

Use Jacobi iteration method for example 5.5 to find the numerical solution of heat equation and compare the result obtains in FS method and iteration method and find the error with error tolerate 0.001. Plot temperature distribution in rectangular R.

Solution: The iteration scheme is

$${u_{ij}}^{k+1} = \frac{{u^k}_{i+1,j}+{u^k}_{i-1,j}+{u^k}_{i,j+1}+{u^k}_{i,j-1}}{4}$$

MATLAB code given in appendix L is used to find the numerical solution of heat equation by Jacobi iterative method and also used to plot the graph of temperature distribution in rectangular region. Numerical solution given in table 7 and and graphical solution is presented in the Figure 29, Figure 30, Figure 31.

Table 7. *Numerical result of heat equation by Jacobi iterative method.*

x/y	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	
0	0	0	0	0	0	0	0	0	0	0	0	
0.1	0	2.630	4.969	6.780	7.919	8.303	7.9187	6.7798	4.9692	2.6306	2	0
0.2	0	5.6462	10.635	14.4749	16.864	17.671	16.864	14.475	10.635	5.646	0	
0.3	0	9.4867	17.781	24.058	27.924	29.217	27.924	24.058	17.7811	9.4866	0	
0.4	0	14.7567	27.375	36.674	42.253	44.1108	42.2533	36.674	27.3748	14.757	0	
0.5	0	22.424	40.804	53.686	61.1403	63.556	61.140	53.686	40.8043	22.424	0	
0.6	0	34.4067	60.2279	76.8409	85.867	88.7160	85.867	76.841	60.228	34.408	0	
0.7	0	55.205	89.3198	108.1853	117.515	120.3193	117.5152	108.185	89.310	55.20	0	
0.8	0	97.275	133.987	149.537	156.218	158.115	156.217	149.537	133.988	97.27	0	
1	200	200	200	200	200	200	200	200	200	200	0	

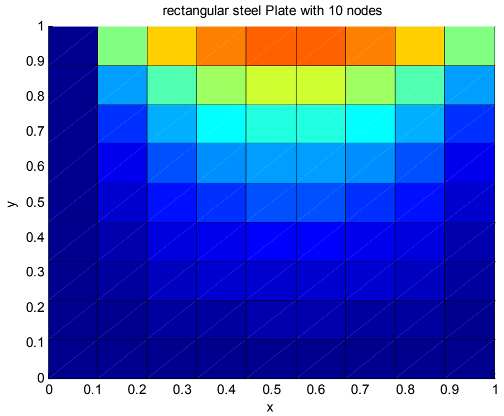

Figure 29. *Temperature distribution in Steel plate iterative method with 10 nodes.*

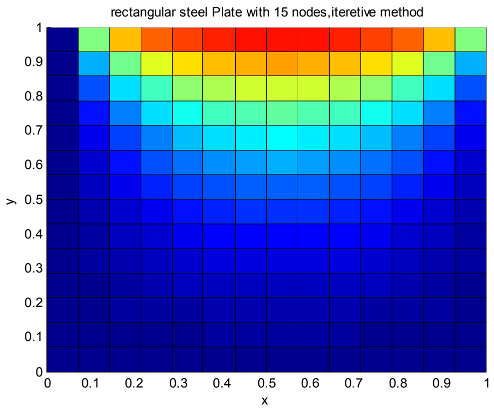

Figure 31. *Temperature distribution in steel plate iterative method with 15 nodes.*

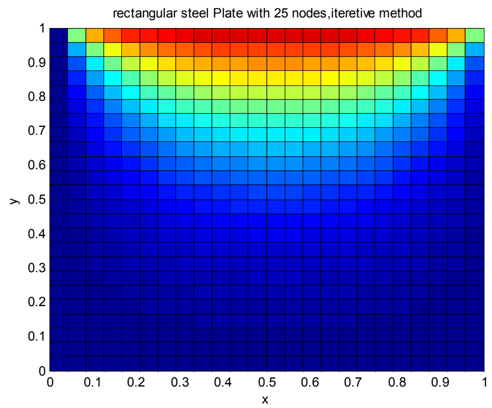

Figure 30. *Temperature distribution inSteel plate iterative method with 25 nodes.*

Discussion:

From Table 6 or Table 7 the above Figures we can see that the solutions obtained in both Fourier series and Jacobi iterative methods are nearly the same, but there is some difference between in the approximation solutions. In Fourier series method we need ten terms of partial sum of Fourier series to be converge to actual solution. However, In the case of Jacobi iterative method we required some number of iterative to converge to exact solution and the accuracy is increased with increasing number of iterative in the calculation. As we observe from Figure 29, Figure 30, Figure 31. The graphical solution is obtained with 10, 15 and 25 nodes with error 0.2448, 0.0928 and 0.0286 respectively, because number of iterative is directly relate to mesh point. In this we can understand that by taking large number of mesh points or nodes which gives minimum error, as the result the numerical solution obtained is more accuracy. Therefore as we compare the two methods one can conclude that Fourier series method is more effective than Jacobi iterative method.

6. Summary and Conclusion

Approximation of functions using Fourier series is an infinite sum of sine and cosine terms. From the disccussion we can observed that functions approximation by fourier series is almost exact when compare to actual function. The approximate function is determined by the coefficients of the trigonometric polynomial and the accuracy increase as in the coefficients of trigonometric polynomial increase. And we also observe that better approximation is obtained by increasing terms of partial sums of Fourier series. Fourier series is used to find the solution of PDEs. Solution to the heat equation is obtained by Fourier series and BTCS method and results are compared. We find that both methods are almost the same but, small difference has been observed between them, due to the truncation error occurred in implicit scheme. We also obtain the numerical solution of Laplace equation using Fourier series and the result is compare with Jacobi iterative method.

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