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Construction of Some Resolvable *t*-designs

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Abstract: The A *t*-design is a generation of balanced incomplete block design (BIBD) where λ is not restricted to the blocks in which a pair of treatments occurs but to the number of blocks in which any *t* treatments (t = 2,3…) occurs. The problem of finding all parameters (t, v, k, λ_t) for which t-(v, k, λ_t) design exists is a long standing unsolved problem especially with λ=1 (Steiner System) as no Steiner *t*-designs are known for $t \geq 6$ when $v > k$. In this study t-design is constructed by relating known BIB designs, combinatorial designs and algebraic structures with t-designs. Additionally, an alternative approach for the construction of *t*-designs that provides a unified framework is also presented.

Keywords: Block Designs, Resolvable Designs, *t*-designs

1. Introduction

A $t-(v,k,\lambda_t)$ design is an incidence structure of points and blocks with the following properties; there are v points, each block is incident with k points, any point is incident with λ_1 blocks, and any t points are incident to λ_t common blocks. Where v, k and λ_t are all positive integers and $v \geq k \geq t$. The four numbers t, v, k and λ_t determine b (blocks) and λ_1 and four numbers themselves cannot be chosen arbitrarily [2].

The incidence structure associated with a t −design can be represented by a matrix. The point-block incidence matrix A, associated with a $t-(v,k,\lambda_t)$ design with b blocks is a $(0-1)$ matrix of v rows and b columns. The elements of A are a_{ij} where i is the point, j is the block and

$$a_{ij} = \begin{cases} 1\, if\, i \in j \\ 0\ otherwise \end{cases}$$

There is a generalization of Fisher's inequality to t −designs which is due to Ray-Chaudhuri [14] and Wilson [16]. If a $t-(v,k,\lambda_t)$ design exists, where $t = 2s$ is even, then the number of blocks $b \geq \binom{v}{s}$. A $t-(v,k,\lambda_t)$ design in which $\lambda = 1$ is called Steiner system. For example a $2-(v,3,1)$ is a Steiner triple system (STS) and a $3-(v,4,1)$ design is a Steiner quadruple system (SQS). A $2-(v,k,\lambda)$ design is called a balanced incomplete block design (BIBD). A t-design is said to have repeated blocks if there are two blocks incident with the same set of k points. A t-design with no repeated blocks is said to be simple [14].

A $t-(v,k,\lambda_t)$ design with $t \geq 3$ are known for only a few values of $v, k$and λ_t. For $t = 3$ there are several infinite families known. For instance, for any prime power q and for any $d \geq 2$, there exists a $3-(q^d+1, q+1,1)$ design known as inversive geometry [7]. When $d = 2$, these designs are known as inversive planes. A Steiner quadruple system $3-(v,4,1)$ is also known to exist for all $v \equiv 2\ or\ 4\ mod\ 6$. Some simple$t$ −designs, have been constructed for $t \leq 5$. Construction of a $6-(v,k,1)$ design remains one of the outstanding open problems in the study of t-designs. Even for $t = 4$and $t = 5$, only a few examples of $t-(v,k,1)$ designs are known. In this study we construct some t −designs, with much emphasis on ≥ 3, $\lambda_t \geq 1$ by identifying BIB designs which are also t −designs [5].

2. Literature Review

The main problem in $t-$designs is the question of existence and the construction of those solutions, given admissible parameters. That is, finding all parameters (t,v,k,λ_t) for which $t-(v,k,\lambda_t)$ design exists. There are many known Steiner $2-$designs but constructing Steiner $t > 2$ it has proved to be much harder. In the case of $t = 3$, Kageyama [11] has shown thatthere is $3-(v,4,1)$ design if andonly if the necessary arithmetic conditions are satisfied. But for larger k, even $k = 5$, the result is far from complete.

For $\geq 3, k \geq 5$ the problem is wide open. All these constructions bear a distinct algebraic flavor in the sense that the underlying set upon which the design is constructed has a nice algebraic structure. Algebraic construction requires that a certain fixed (big) group to act as a group of automorphisms for the desired design.

Mathon and Rosa [12] came up with block spreading method for $t = 2$ and for prime power index. Let v be a positive integer $v \geq 2$ and let q be a prime power. Suppose that there exists a $S_q(2, k, v)$ design satisfying $q \geq v + 1$. Then there exists a group divisible design (GDD) of group type $(q^d)^v$ with block size k and index one, whenever $d \geq \binom{v}{2}$. This method has application in the construction of Steiner 2 −designs.

Let v and t be positive integers, $2 \leq t \leq v$, and let q be a prime power. Then there exists a number $q_0 = q_0(t, v)$ such that for any $S_q(t, k, v)$ design satisfying $q \geq q_0$, there is a t −GDD of group type $(q^d)^v$ with block size k and index one whenever $d \geq \binom{v}{t}$. Let v, t and λ be a positive integers $2 \leq t \leq v$. Then there exists a number $q_0 = q_0(t, v)$ such that for any $S_\lambda(t, k, v)$ design with prime power decomposition $\lambda = q_1, q_2, q_3 \cdots q_n$ satisfying $q_i \geq q_0$; $1 \leq i \leq n$; there is a t −GDD of group type $(\lambda^d)^v$ with block size k and index one whenever $d \geq \binom{v}{t}$. This generalized "block spreading" construction has several application such as constructing new Steiner 3 −designs and new group divisible t −designs with index one. Limitation of this method is that the bounds on d are too large.

A block design is a family of b subsets of a set S of v elements such that, for some fixed k and λ, with $k < v, \lambda > 0$; each subset has k elements, and each pair of elements of t occurs together in exactly λ subsets. The elements of S are called varieties, and the subsets of S are called the blocks. From Anderson [2], in a block design each element lies in exactly r blocks, where

$$r(k-1) = \lambda(v-1) \text{and} bk = vr \qquad (1)$$

The five parameters b, v, r, k, λ of a block design are therefore not independent, but have two restrictions as stated in the theorem. Whatever b, v, r, k, λ are, they must satisfy (1), but conversely if five numbers b, v, r, k, λ satisfy (1.1), there is no guarantee that a (b, v, r, k, λ) − configuration exists as described by [15].

Mohácsy and Ray-Chaudhuri [12] constructed t −designs from known t − wise balanced designs. In his works he showed that, given a positive integer k and a $t - (v, (k_1, k_2 \cdots k_s), \lambda)$ design D, with all blocks-sizes k_i occurring in D and $1 \leq t \leq k \leq k_1 \leq k_2 \cdots \leq k_s$, the construction produces a $t - (v, k, n\lambda)$ design D^*, with $n = L.C.M\left[\binom{k_1-t}{k-t}, \cdots \binom{k_s-t}{k-t}\right]$. Onyango [16] on his part constructed t −designs with $t = 3$ and $\lambda = 1$ from balanced incomplete block design.

Incidence Matrix

The incidence matrix A of a (v, b, r, k, λ)-BIBD satisfies the following properties: every column of A contains exactly k "1"s; every row of A contains exactly r "1"s; and two distinct rows of A contain "1" in exactly λ columns

Theorem (Stinson (2004)). Let A be a $v \times b$0-1 matrix and let $2 \leq k < v$. Then A is an incidence matrix of a(v, b, r, k, λ)-BIBD if and only if $AA' = \lambda J_v + (r - \lambda)I_v$and $u_v A = k u_b$ where I_v denotes a $v \times v$ unity matrix and J_v denotes a $v \times v$ matrix with every entry equal to 1.

Example of constructing BIB designs

Now consider $v = 7, b = 7, k = 3$, the conditions are fulfilled with;

$$r = \frac{bk}{v} = \frac{7 \times 3}{7} = 3$$

$$\lambda = r\frac{(k-1)}{v-1} = \frac{3 \times 2}{6} = 1$$

Let the points be A, B, C, D, E, F, and G. Ordering the 3 blocks with A first and assume that B-C, D-E and F-G are together in these blocks. The following results are obtained:

Table 1. *Results of Step one.*

A	A	A
B	D	F
C	E	G

Second step

Next B and C must occur twice more and not together (order of blocks not important). The result is:

Table 2. *Results of Step two.*

A	A	A	B	B	C	C
B	D	F				
C	E	G				

Third step

D and F must occur together. We can assume this happens in a B block. The E and G must be together in the other B block. Then there is only one choice for the two C blocks (because order is unimportant). Results obtained are:

Table 3. 2 − (7,3,1)-*configuration.*

A	A	A	B	B	C	C
B	D	F	D	E	D	E
C	E	G	F	G	G	F

When using this design for practical experiments randomization is a must. Treatments according to the labels will be randomized as per the order of the blocks, and the order of the three treatments within a block. The complementary design is constructed by replacing each block with a block consisting of the remaining points. For this case this results in:

Table 4. *Complementary design.*

D	B	B	A	A	A	A
E	C	C	C	C	B	B
F	F	D	E	D	E	D
G	G	E	G	F	F	G

Results obtained are $v = 7, r = 4, b = 7, k = 4$ and $\lambda = 2$. It is noted that the same BIB designs can be constructed by use of PG (2, 2).

3. Construction of Resolvable $3-(v,k,\lambda_3)$ Design

In this construction, technique introduced by Adhikari [1] of using the symmetric differences of pairs of blocks of incomplete blocks designs to construct other designs and the technique of arithmetic of integers modulo *n*is applied.

3.1. Resolvable 3-design with Parameters $v=8, b=14, r=7, k=4, \lambda_2=3, \lambda_3=1$

Consider resolvable 3-design with parameters $v=8, b=14, r=7, k=4, \lambda_2=3, \lambda_3=1$

Let $v=\{0,1,2,3,4,5,6,\infty\}$ be the set of equivalence classes mod 7 and ∞ and $\beta=\{(1,3,4,\infty),(0,2,5,6)\}$ form a base for a (14, 8, 7, 4, 3) $\infty-$ cyclic design mod 7. When2 is added to each element of $\{1,3,4,\infty\}$ and $\{0,2,5,6\}$ and same process is continued, blocks of the design are obtained as follows;

$\{1,3,4,\infty\}$	$\{0,2,5,6\}$
$\{2,5,6,\infty\}$	$\{2,4,0,1\}$
$\{5,0,1,\infty\}$	$\{4,6,2,3\}$
$\{0,2,3,\infty\}$	$\{6,1,4,5\}$
$\{2,4,5,\infty\}$	$\{1,3,6,0\}$
$\{4,6,0,\infty\}$	$\{3,5,1,2\}$
$\{6,1,2,\infty\}$	$\{5,0,3,4\}$

Replacing residues with integers and ∞ with 8, the following results are obtained;

$\{1,3,4,8\}\{2,5,6,7\}$

$\{3,5,6,8\}\{1,2,4,7\}$

$\{5,7,1,8\}\{1,4,5,6\}$

$\{7,2,3,8\}\{1,4,5,6\}$

$\{2,4,5,8\}\{1,3,6,7\}$

$\{4,6,7,8\}\{1,2,3,5\}$

$\{6,1,2,8\}\{3,4,5,7\}$

Computing the differences modulo 7 and ∞from pairs of distinct elements in β, the following values of the block designs are obtained:

$$\beta=\{(1,3,4,\infty),(0,2,5,6)\}$$

3-1 = 2	1-3 = 5	2-0 = 2	0-2 = 5
4-1 = 3	1-4 = 4	5-0 = 5	0-5 = 2
∞-1 = ∞	1-∞ = -∞	6-0 = 6	0-6 = 1
4-3 = 1	3-4 = 6	5-2 = 3	2-5 = 4
∞-3 = ∞	3-∞ = -∞	6-2 = 4	2-6 = 3
∞-4 = ∞	4-∞ = -∞	6-5 = 1	5-6 = 6

This results in ∞, -∞ and each non zero residue mod 7 exactly thrice as a difference of two elements in β. This design is a (14,8,7,4,3)- BIBD which would result into a 3-(8,4,1) design.

3.2. Construction of t-design with Parameters $v=12, b=22, r=11, k=6, \lambda_2=5, \lambda_3=2$

Affine 3-design with parameters $v=12, b=22, r=11, k=6, \lambda_2=5, \lambda_3=2$

Let $v=\{0,1,2,3,4,5,6,7,8,9,10,\infty\}$ is the set of equivalence classes mod 11 and ∞ and $\beta=\{(1,3,4,5,9,\infty),(0,2,6,7,8,10\}$ then β is a base for the design.

$\{1,3,4,5,9,\infty\}\{0,2,6,7,8,10\}$

$\{3,5,6,7,0,\infty\}\{2,4,8,9,10,1\}$

$\{5,7,8,9,2,\infty\}\{4,6,10,0,1,3\}$

$\{7,9,10,0,4,\infty\}\{6,8,1,2,3,5\}$

$\{9,0,1,2,6,\infty\}\{8,10,3,4,5,7\}$

$\{0,2,3,4,8,\infty\}\{10,1,5,6,7,9\}$

$\{2,4,5,6,10,\infty\}\{1,3,7,8,9,0\}$

$\{4,6,7,8,1,\infty\}\{3,5,9,10,0,2\}$

$\{6,8,9,10,3,\infty\}\{5,7,0,1,2,4\}$

$\{8,10,0,1,5,\infty\}\{7,9,2,3,4,6\}$

$\{10,1,2,3,7,\infty\}\{9,0,4,5,6,8\}$

Replacing residues with integers and ∞ with 12, the following results are obtained;

$\{1,3,4,5,9,12\}\{11,2,6,7,8,10\}$

$\{3,5,6,7,11,12\}\{2,4,8,9,10,1\}$

$\{5,7,8,9,2,12\}\{4,6,10,11,1,3\}$

$\{7,9,10,11,4,12\}\{6,8,1,2,3,5\}$

$\{9,11,1,2,6,12\}\{8,10,3,4,5,7\}$

$\{11,2,3,4,8,12\}\{10,1,5,6,7,9\}$

$\{2,4,5,6,10,12\}\{1,3,7,8,9,11\}$

$\{4,6,7,8,1,12\}\{3,5,9,10,11,2\}$

$\{6,8,9,10,3,12\}\{5,7,11,1,2,4\}$

$\{8,10,11,1,5,12\}\{7,9,2,3,4,6\}$

$\{10,1,2,3,7,12\}\{9,11,4,5,6,8\}$

Case of $3-(8,4,2)$

Construction of $3-(8,4,2)$ where $v=8, b=28, r=14, k=4, \lambda_{2,}=6, \lambda_3=2$

If $v=\{0,1,2,3,4,5,6,\}$ is the set of equivalence classes mod 7 and ∞ , and $\beta=\{(1,3,4,\infty),(0,2,5,6),(1,2,5,\infty),0,3,4,6)\}$

Following the same procedure, design below with integers as elements is obtained;

{1,3,4,8}{2,5,6,7}

{3,5,6,8}{1,2,4,7}

{5,7,1,8}{1,4,5,6}

{7,2,3,8}{1,4,5,6}

{2,4,5,8}{ 1,3,6,7}

{4,6,7,8}{1,2,3,5}

{6,1,2,8}{3,4,5,7}

{1,5,6,8}{2,3,4,7}

{3,7,1,8}{4,5,6,2}

{5,2,3,8}{6,7,1,4}

{7,4,5,8}{1,2,3,6}

{2,6,7,8}{ 3,4,5,1}

{4,1,2,8}{5,6,7,3}

{6,3,4,8}{7,1,2,5}

Construction of $3-(8,4,3)$ design where $b=42, v=8, k=4, r=21, \lambda_2=9, \lambda_3=3$

If $v=(0,1,2,3,4,5,6,\infty)$ is the set of equivalence classes mod 7 and ∞ and $\beta=\{(1,3,4,\infty),(0,2,5,6,),(1,2,5,\infty),(0,3,4,6),(0,3,5,6),(1,2,4,\infty)\}$

Using the same procedure, the design below with integers as elements is obtained.

(1,3,4,8)	(2,5,7,6)	(1,5,6,8)
(3,5,6,8)	(4,7,2,1)	(3,7,1,8)
(5,7,1,8)	(6,2,4,3)	(5,2,3,8)
(7,2,3,8)	(1,4,6,5)	(7,4,5,8)
(2,4,5,8)	(3,6,1,7)	(2,6,7,8)
(4,6,7,8)	(5,1,3,2)	(4,1,2,8)
(6,1,2,8)	(7,3,5,4,)	(6,3,4,8)
(2,3,4,7)	(7,3,5,6)	(1,2,4,8)
(4,5,6,2)	(2,5,7,1)	(3,4,6,8)
(6,7,1,4)	(4,7,2,3)	(5,6,1,8)
(1,2,3,6)	(6,2,4,5)	(7,1,3,8)
(3,4,5,1)	(1,4,6,7)	(2,3,5,8)
(5,6,7,3)	(3,6,1,2)	(4,5,7,8)
(7,1,2,5)	(5,1,3,4)	(6,7,2,8)

Case of $3-(12,6,4)$

Construction of $3-(12,6,4)$ where $b=44, v=12, r=22, k=6, \lambda_2=10, \lambda_3=4$

Let $v=\{0,1,2,3,4,5,6,7,8,9,10,\infty\}$ is the equivalence classes' mod 11 and ∞ and $\beta=\{(1,3,4,5,9,\infty),(2,6,7,8,10,0),(0,3,7,8,9,\infty),(1,24,5,6,10)\}$ then β is the base for the design below with integers as elements:

(1,3,4,5,9,12)	(2,6,7,8,10,11)
(3,5,6,7,11,12)	(4,8,9,10,1,2)
(5,7,8,9,2,12)	(6,10,11,1,3,4)
(7,9,10,11,4,12)	(8,1,2,3,5,6)
(9,11,1,2,6,12)	(10,3,4,5,7,8)
(11,2,3,4,8,12)	(1,5,6,7,9,10)
(2,4,5,6,10,12)	(3,7,8,9,11,1)
(4,6,7,8,1,12)	(5,9,10,11,2,3)
(6,8,9,10,3,12)	(7,11,1,2,4,5)
(8,10,11,1,5,12)	(9,2,3,4,6,7)
(10,1,2,3,7,12)	(11,4,5,6,8,9)
(3,7,8,9,11,12)	(1,2,4,5,6,10
(5,9,10,11,2,12)	(3,4,6,7,8,1)
(7,11,1,2,4,12)	(5,6,8,9,10,3)
(9,2,3,4,6,12)	(7,8,10,11,1,5)
(11,4,5,6,8,12)	(9,10,1,2,3,7)
(2,6,7,8,10,12)	(11,1,3,4,5,9)
(4,8,9,10,1,12)	(2,3,5,6,7,11)
(6,10,11,1,3,12)	(4,5,7,8,9,2)
(8,1,2,3,5,12)	(6,7,9,10,11,4)
(10,3,4,5,7,12)	(8,9,11,1,2,4)
(1,5,6,7,9,12)	(10,11,2,3,4,6)

This construction is equivalent to "sum construction ", of BIBDs, but in this case a BIBD is added to a BIBD that is automorphic to it. Therefore new BIBDs can be formed by the collection of a BIBD with its automorphic BIBDs.

4. Conclusion

The study has presented an alternative method that is simpler and unified for the construction of BIBDs that are very important in the experimental designs. As it provides designs for different values of k, unlike many methods that provide designs for a single value of k. The construction framework designed provides a platform at which new BIBDs can be formed by the collection of a BIBD with its automorphic BIBDs. In order to obtain combinatorial constructions of unique block designs, different kind of combinatorial designs are very effective.

Recommendations

Although this study has provided a technique for the construction of $t-$ designs, it is still clear that construction method of $t-$designs is not known in general. In fact, it is not clear how one might construct $t-$designs with arbitrary block size.

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