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Strong Solutions of Navier-Stokes-Poisson Equations for Compressible Non-Newtonian Fluids

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Abstract: This paper study the Navier-Stoke-Poisson equations for compressible non-Newtonian fluids in one dimensional bounded intervals. The motion of the fluid is driven by the compressible viscous isentropic flow under the self-gravitational and an external force. The local existence and uniqueness of strong solutions was proved based on some compatibility condition. The main condition is that the initial density vacuum is allowed.

Keywords: Strong Solutions, Navier-Stokes-Poisson Equations, Non-Newtonian Fluids, Vacuum

1. Introduction

In this paper we study a class of one-dimensional isentropic compressible non-Newtonian fluids of Navier Stokes Poisson system:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + \rho\Phi_x - ((u_x^2+\mu_0)^{\frac{p-2}{2}} u_x)_x + P_x = \rho f, \\ \Phi_{xx} = 4\pi g(\rho - \frac{1}{|\Omega|}\int_\Omega \rho dx), \end{cases} \quad (1)$$

With the initial and boundary conditions

$$\begin{cases} (\rho, u, \Phi)(x,0) = (\rho_0, u_0, \Phi_0), \quad x \in [0,1] \\ u(0,t) = u(1,t) = 0, \ \Phi(0) = \Phi(1) = 0, \quad t \in [0,T] \end{cases} \quad (2)$$

Where $(x,t) \in \Omega_T, \Omega_T = I \times (0,T), I = (0,1).$

$\rho, u, \Phi, P \equiv a\rho^\gamma, a > 0, \gamma > 1$ denote the unknown density, velocity, geopotential and pressure, respectively.

In the sense of physics, the motion of the fluid is driven by the compressible viscous isentropic flow under the self-gravitational and an external force f, the initial density $\rho_0 \ge 0, \mu_0 > 0$ are given constants.

During the past decades, fluid dynamics has attracted the attention of many mathematicians and engineers. The study of non-Newtonian fluid mechanics is of great significant because of the non-Newtonian fluids are widely used as up to date ones in various fields of applied sciences, such as the models for the flow of glacier, the flow of blood through arteries be proposed in blood rheology, the dynamics of tectonic plates in the earth's mantle in geology etc. ([1-3]).

Up to now, the results on non-Newtonian fluids are quite few. In [4], the local existence and uniqueness results of non-Newtonian fluids were given in the case of $\rho_0 \ge 0, 1 < p < 2$ and by assuming a similar compatibility condition as (3). [5] studied the global existence and uniqueness results of heat-conducting fluids if $p > 2$ and the initial density in H^1 norm is small enough. The results on fluid particle interaction non-Newtonian models, see [6-7].

For the Newtonian fluids without considering the energy consevation equation term have been studied by many authors ([8-15]). For detail, [9] applied the weak convergence method showed the existence of global weak solutions under the assumption that $\gamma \ge 3/2$ if $n = 2$ and $\gamma \ge 9/5$ if $n = 3$. Later, this method was improved to reduce more general results ([10-12]). In [13-15], we can find some local existence results on strong solutions in three dimensional space followed the compatibility condition

$$-\mu\Delta u_0 + \nabla P(\rho_0) = \rho_0 g^{1/2} \text{ for some } g \in L^2(\Omega) \quad (3)$$

As for the Navier-Stokes-Poisson equations for Newtonian fluids with density dependent or independent with viscosity, the existence of strong solutions, regularity and large time behavior of solutions were investigated, for these results, see [16-20]. In this paper, we discuss the system (1)-(2) with $\rho_0 \geq 0, \mu_0 \geq 0, p > 2$, we prove the local existence and uniqueness of strong solutions under some conditions. As we know that when $p > 2$, the second equation of (1) is always with degeneration. Moreover, the initial density is allowed with vacuum and the strong nonlinearity of equations bring us another difficulty.

2. Main Results

2.1. Main Theorem

Theorem 2.1.1 Assume that (ρ_0, u_0, f) satisfies the following conditions

$0 \leq \rho_0 \in H^1(I), u_0 \in H_0^1(I) \cap H^2(I), f \in L^\infty(0.T; L^2(I)),$ $f_t \in L^\infty(0.T; L^2(I))$, and if there is a function $g \in L^2(I)$, such that the initial data satisfy the following compatibility condition:

$$-[(u_{0x}^2 + \mu_0)^{(p-2)/2} u_{0x}]_x + P_x(\rho_0) = \rho_0^{1/2} g \quad \text{for a.e. } x \in I \tag{4}$$

Then there exist a time $T_* \in (0, +\infty)$ and a unique strong solution (ρ, u, Φ) to (1)-(2) such that

$$\begin{cases} \rho \in C([0.T_*]; H^1(I)), \rho_t \in C([0.T_*]; L^2(I)), \\ u \in C([0.T_*]; H_0^1(I)) \cap L^\infty(0, T_*; H^2(I)), \\ u_t \in L^2([0.T_*]; H_0^1(I)), \sqrt{\rho} u_t \in L^\infty(0, T_*; L^2(I)), \\ [(u_x^2 + \mu_0)^{(p-2)/2} u_x]_x \in C([0, T_*]; L^2(I)), \\ \Phi \in L^\infty(0.T_*; H^3(I)), \Phi_t \in L^\infty(0.T_*; H^2(I)). \end{cases}$$

2.2. Preliminaries

First of all, some known facts are given for latter use.

Lemma 2.2.1 Assume that $f = 0$ on $\partial\Omega$, where $\Omega \subset R^1$ is bounded and open, $f \in C^{2+\alpha}(\overline{\Omega}), 0 < \alpha < 1$. Then

$$|f'|_{L^\infty(\Omega)} \leq d^{1/2}(\Omega) |f''|_{L^2(\Omega)}$$

Where $d(\Omega)$ denotes the length of Ω.

Lemma 2.2.2 Let H be a Hilbert space with a scalar product $(.,.)_H$ and let X be a Banach space such that $X \hookrightarrow H \cong H^* \hookrightarrow X^*$ and X is dense in H, $p > 1$. Then

$$W \equiv \{u \in L^p(I; X); \frac{du}{dt} \in L^{p'}(I; X^*)\} \hookrightarrow C(I; H)$$

3. Existence of Solutions

In this section, we will prove the local existence of strong solutions. To get the existence of strong solutions, some more regular estimates are required. Provide that (ρ, u, Φ) is a smooth solutions of (1)-(2) and $\rho_0 \geq \delta$, where $0 < \delta \leq 1$ is a positive number, as we can deal with approximate system, we only consider initial nonvacuum. Combining the classical results of $(1)_3$ with our correlated uniform estimates, we can get the existence of strong solutions of our system. Throughout the paper, we denote by

$$M_0 = 1 + \mu_0 + \mu_0^{-1} + |\rho_0|_{H^1} + |g|_{L^2} + |f|_{L^\infty(0,T;L^2(I))} + |f_t|_{L^\infty(0,T;L^2(I))}$$

In the following sections, we will use simplified notations for standard Sobolev spaces and Bochner spaces, such as $L^p, H^1, C[0,1; H^1]$ etc.

● A priori Estimates for Smooth Solutions

We construct an anxiliary function

$$\Psi(t) = \sup_{0 \leq s \leq t}(1 + |u(s)|_{W_0^{1,p}} + |\rho(s)|_{H^1} + |\sqrt{\rho} u_t(s)|_{L^2})$$

Then we will prove that $\Psi(t)$ is local bounded (in time). Next, each terms of $\Psi(t)$ will be estimated as follows:

● Estimate for $|\rho|_{H^1}$

Firstly, By $(1)_2$

$$[((u_x)^2 + \mu_0)^{(p-2)/2} u_x]_x = \rho u_t + \rho u u_x + \rho \Phi_x + P_x - \rho f \tag{5}$$

Then

$$|u_{xx}| \leq C |\rho u_t + \rho u u_x + \rho \Phi_x + P_x - \rho f|$$

Taking it by L^2-norm, Young's inequality, we get

$$|u_{xx}|_{L^2} \leq C |\rho u_t|_{L^2} + |\rho u u_x|_{L^2} + |\rho \Phi_x|_{L^2} + |P_x|_{L^2} + |\rho f|_{L^2}$$

$$\leq C |\rho|_{L^\infty}^{1/2} |\sqrt{\rho} u_t|_{L^2} + |\rho|_{L^\infty} |u|_{L^\infty} |u_x|_{L^p} + |\rho|_{L^\infty} |\Phi_x|_{L^2} + |P_x|_{L^2} + |\rho f|_{L^2} \tag{6}$$

We deal with the term of $|\Phi_x|_{L^2}$, Multiplying $(1)_3$ by Φ and integrating over (0, 1), we get

$$-\int_0^1 \Phi_{xx} \Phi dx = -4\pi g (\int_0^1 \rho \Phi dx - m_0 \int_0^1 \Phi dx)$$

$$\leq 8\pi g m_0 |\Phi|_{L^\infty} \leq 1/2 |\Phi_x|_{L^2}^2 + 32\pi^2 g^2 m_0^2$$

Consequently,

$$\int_0^1 \Phi_x^2 dx \leq C (\int_0^1 \rho dx)^2 \leq C(m_0) \tag{7}$$

where $m_0 = \int_0^1 \rho_0(x) dx > 0$ is the initial mass.

Substituting (7) into (6), we get

$$|u_{xx}(t)|_{L^2} \leq C \Psi^{\gamma+2}(t) \tag{8}$$

Then muntiplying $(1)_1$ by ρ, integrating over (0, 1) with respect to x, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |\rho|^2\ ds+\int_0^1(\rho u)_x\rho dx=0$$

Integrating by parts, using Sobolev inequality, we deduct that

$$\frac{d}{dt}|\rho(t)|_{L^2}^2\le|u_{xx}|_{L^2}|\rho|_{L^2}^2 \qquad (9)$$

Then differential $(1)_1$ with respect to x, and multiplying it by ρ_x, integrating over (0, 1) 0n x, and using Sobolev inequality, we have

$$\frac{d}{dt}|\rho_x|^2\ dx=-\int_0^1[\frac{3}{2}u_x(\rho_x)^2+\rho\rho_x u_{xx}](t)dx$$

$$\le\frac{3}{2}[|u_x|_{L^\infty}|\rho_x|_{L^2}^2+|\rho|_{L^\infty}^2|u_{xx}|_{L^2}]$$

$$\le 3|\rho_x|_{L^2}^2|u_{xx}|_{L^2} \qquad (10)$$

From (9) and (10), by Gronwall's inequality, it follows that

$$\sup_{0\le t\le T}|\rho(t)|_{H^1}^2\le|\rho_0|_{H^1}^2\ \exp\{4\int_0^t|u_{xx}|_{L^2}ds\}$$

$$\le C\exp(C\int_0^t\Psi^{\gamma+2}(s)ds) \qquad (11)$$

And using $(1)_1$ we can also obtain

$$|\rho_t(t)|_{L^2}\le|\rho_x(t)|_{L^2}|u(t)|_{L^\infty}+|\rho(t)|_{L^\infty}|u_x(t)|_{L^2}$$

$$\le C\Psi^2(t) \qquad (12)$$

Where C is a positive constant, depending only on M_0.

- Estimate for $|u|_{W_0^{1,p}}$

Multiplying $(1)_2$ by u_t, and integrating over Ω_t, we have

$$\int_0^t\int_0^1\rho|u_t|^2\ dxds+\frac{1}{2}\frac{d}{dt}\int_0^1(\int_0^{u_x^2}(s+\mu_0)^{(p-2)/2}ds)dx$$
$$=\int_0^1 Pu_x(t)dx-\int_0^1 Pu_x(0)dx+\int_0^t\int_0^1[(\rho f-\rho uu_x$$
$$-\rho\Phi_x)u_t-P_t u_x]dx \qquad (13)$$

Since

$$\int_0^{u_x^2(t)}(s+\mu_0)^{(p-2)/2}ds=\frac{2}{p}[((u_x)^2+\mu_0)^{p/2})-\mu_0{}^{p/2}]$$

$$\ge\frac{2}{p}(|u_x|^p-\mu_0{}^{p/2}) \qquad (14)$$

and

$$\int_0^{u_x^2(0)}(s+\mu_0)^{(p-2)/2}ds\le\int_0^{u_x^2(0)}(s^{(p-2)/2}+\mu_0)ds$$

$$\le\mu_0 u_x^2(0)+\frac{2}{p}|u_x(0)|^p \qquad (15)$$

Substituting (14), (15) into (13), by $(1)_1$, Sobolev inequality and Young's inequality, we have

$$\int_0^t|\sqrt{\rho}u_t(s)|_{L^2}^2ds+|u_x(t)|_{L^p}^p\le C+\int_0^t\int_0^1(|\rho fu_t|$$
$$+|\rho uu_x u_t|+|\rho\Phi_x u_t|)dxds+\int_0^1|Pu_x|dx$$
$$+\int_0^t\int_0^1(|P_x uu_x|+\gamma|Pu_x u_x|)dxds$$
$$\le C+C_\eta\int_0^t|\sqrt{\rho}f(s)|_{L^2}^2ds+C_\eta\int_0^t|\sqrt{\rho}(s)|_{L^\infty}|u_x(s)|_{L^p}^2$$
$$|u_x(s)|_{L^p}^2\ ds+C_\eta\int_0^t|\rho|_{L^2}|\Phi_x|_{L^2}^2\ ds+\int_0^t(|P_x(s)|_{L^2}|u(s)|_{L^\infty}$$
$$|u_x(s)|_{L^p}+\gamma|P(s)|_{L^\infty}|u_x(s)|_{L^p}|u_x(s)|_{L^p})ds+C|P(t)|_{L^2}^2$$
$$+\frac{1}{2}\int_0^t|\sqrt{\rho}u_t(s)|_{L^2}^2\ ds+\frac{1}{2}|u_x(t)|_{L^p}^p \qquad (16)$$

Where $0<\eta\le1$, to estimate (16), combining with $(1)_1$ the following estimates are hold

$$|\rho(t)|_{L^\infty}+|P(t)|_{H^1}\le|\rho(t)|_{H^1}+C|\rho(t)|_{L^\infty}^{\gamma-1}+|\rho(t)|_{H^1}\le C\Psi^\gamma(t)$$

$$\int_0^1|P(t)|^2\ dx=\int_0^1|P(0)|^2\ dx+\int_0^t\frac{\partial}{\partial s}(\int_0^1(P(s))^2dx)ds$$

$$\le\int_0^1|P(0)|^2\ dx+2\int_0^t\int_0^1 P(-P_x u-\gamma Pu_x)dxds$$

$$=\int_0^1|P(0)|^2\ dx+2\int_0^t\int_0^1 a\gamma\rho^{\gamma-1}P(-\rho_x u-\rho u_x)dxds$$

$$\le C+C\int_0^1|\rho(s)|_{L^\infty}^{\gamma-1}|P(s)|_{L^\infty}|\rho_x(s)|_{L^2}|u_x(s)|_{L^p}\ ds$$

$$\le C(1+\int_0^1\Psi^{2\gamma+1}(s)ds) \qquad (17)$$

Combining (16), (17), yields

$$\int_0^t|\sqrt{\rho}u_t(s)|_{L^2}^2ds+|u_x(t)|_{L^p}^p\le C(1+\int_0^1\Psi^{2\gamma+1}(s)ds) \qquad (18)$$

Where C is a positive constant, depending only on M_0.

- Estimate for $|\sqrt{\rho}u_t(t)|_{L^2}$

Differentiating $(1)_2$ with respect to t, multiplying it by u_t, integrating it over (0, 1) on x, we derive

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \rho|u_t|^2\,dx+\int_0^1[((u_x)^2+\mu_0)^{(p-2)/2}u_x]_t u_{xt}dx$$
$$=\int_0^1[(f-u_t-uu_x-\Phi_x)\rho_t-\rho u_t u_x-\rho\Phi_{xt}+\rho f_t]u_t dx$$
$$+\int_0^1 P_t u_{xt}dx \tag{19}$$

Note that

$$(((u_x)^2+\mu_0)^{(p-2)/2}u_x)_t u_{xt}{}^2$$
$$=((u_x)^2+\mu_0)^{(p-4)/2}((p-1)(u_x)^2+\mu_0)(u_{xt})$$
$$\geq(\mu_0)^{(p-2)/2}(u_{xt})^2 \tag{20}$$

From (20) and $(1)_1$, (19) can be rewritten into

$$\frac{d}{dt}\int_0^1\rho|u_t|^2\,dx+\int_0^1|u_{xt}|^2\,dx$$
$$\leq\int_0^1 2\rho|u||u_t||u_{xt}|dx+\int_0^1|\rho_x||u|^2|u_x||u_t|dx$$
$$+\int_0^1|\rho_x||u||f||u_t|dx+\int_0^1\rho|u||u_x|^2|u_t|dx$$
$$+\int_0^1\rho|u_x||f||u_t|dx+\int_0^1|P_x||u||u_{xt}|dx$$
$$+\int_0^1\gamma P|u_x||u_{xt}|dx+\int_0^1|\rho_x||u||\Phi_x||u_t|dx$$
$$+\int_0^1|\rho||u_x||\Phi_x||u_t|dx+\int_0^1\rho|u_t||u_x||u_t|dx$$
$$+\int_0^1\rho|u_t||f_t|dx+\int_0^1\rho|\Phi_{xt}||u_t|dx=\sum_{j=1}^{12}I_j \tag{21}$$

Using Sobolev inequality, Young's inequality, (8), we get

$$I_1\leq 2|\rho|_{L^\infty}^{1/2}|u|_{L^\infty}|\sqrt{\rho}u_t|_{L^2}|u_{xt}|_{L^2}\leq C\Psi^5(t)+\frac{1}{10}|u_{xt}|_{L^2}^2$$

$$I_2\leq|\rho_x|_{L^2}|u_x|_{L^p}^2|u_x|_{L^p}|u_{xt}|_{L^2}\leq C\Psi^8(t)+\frac{1}{10}|u_{xt}|_{L^2}^2$$

$$I_3\leq|\rho_x|_{L^2}|u_x|_{L^p}|f|_{L^2}|u_{xt}|_{L^2}\leq C\Psi^4(t)+\frac{1}{10}|u_{xt}|_{L^2}^2$$

$$I_4\leq|\rho|_{L^\infty}|u_x|_{L^p}^3|u_{xt}|_{L^2}\leq C\Psi^8(t)+\frac{1}{10}|u_{xt}|_{L^2}^2$$

$$I_5\leq|\rho|_{L^\infty}|u_x|_{L^p}|f|_{L^2}|u_t|_{L^\infty}\leq C\Psi^4(t)+\frac{1}{10}|u_{xt}|_{L^2}^2$$

$$I_6\leq|P_x|_{L^2}|u|_{L^\infty}|u_{xt}|_{L^2}\leq C\Psi^{2(\gamma+1)}(t)+\frac{1}{10}|u_{xt}|_{L^2}^2$$

$$I_7\leq C|P_x|_{L^2}|u_x|_{L^p}|u_{xt}|_{L^2}\leq C\Psi^{2(\gamma+1)}(t)+\frac{1}{10}|u_{xt}|_{L^2}^2$$

$$I_8\leq|\rho_x|_{L^2}|u_x|_{L^2}|\Phi_x|_{L^2}|u_{xt}|_{L^2}\leq C\Psi^4(t)+\frac{1}{10}|u_{xt}|_{L^2}^2$$

$$I_9\leq|\rho|_{L^\infty}|u_x|_{L^2}|\Phi_x|_{L^2}|u_{xt}|_{L^2}\leq C\Psi^4(t)+\frac{1}{10}|u_{xt}|_{L^2}^2$$

$$I_{10}\leq|\sqrt{\rho}u_t|_{L^2}^2|u_x|_{L^2}|u_{xt}|_{L^2}\leq C\Psi^4(t)+\frac{1}{10}|u_{xt}|_{L^2}^2$$

$$I_{11}\leq|\rho|_{L^\infty}^{1/2}|\sqrt{\rho}u_t|_{L^2}|f_t|_{L^2}\leq C\Psi^{3/2}(t)$$

$$I_{12}\leq|\sqrt{\rho}|_{L^\infty}|\sqrt{\rho}u_t|_{L^2}|\Phi_{xt}|_{L^2}\leq C\Psi^4(t)+|\Phi_{xt}|_{L^2}^2$$

We deal with the estimate of Φ_{xt}.

Differential $(1)_3$ with respect to t, multiplying it by Φ_t and integrating over (0, 1), we have

$$\int_0^1\Phi_{xxt}\Phi_t dx=4\pi g\int_0^1\rho_t\Phi_t dx$$

Then

$$\int_0^1|\Phi_{xt}|^2\,dx\leq C|\rho_t|_{L^2}|\Phi_{xt}|_{L^2}\leq C|\rho_t|_{L^2}^2+\frac{1}{2}|\Phi_{xt}|_{L^2}^2$$

Thus

$$\int_0^1|\Phi_{xt}|^2\,dx\leq C|\rho_t|_{L^2}^2\leq C|\rho_x|_{L^2}|u_x|_{L^p}\leq C\Psi^2(t)$$

Substituting these estimates into (21), we obtain

$$\frac{d}{dt}|\sqrt{\rho}u_t(t)|_{L^2}^2+|u_{xt}|_{L^2}^2\leq C\Psi^{2\gamma+6}+C|\sqrt{\rho}u_t(t)|_{L^2}^2 \tag{22}$$

Then integrating (22) over $(\tau,t)\subset(0,t)$, we deduce that

$$|\sqrt{\rho}u_t(t)|_{L^2}^2+\int_\tau^t|u_{xt}(s)|_{L^2}^2ds\leq C\int_\tau^t\Psi^{2\gamma+6}+C|\sqrt{\rho}u_t(\tau)|_{L^2}^2 \tag{23}$$

We estimate $|\sqrt{\rho}u_t(\tau)|_{L^2}^2$ as follows:

Using $(1)_2$ and according to the smooth of (ρ, u, Φ) we have

$$\int_0^1\rho|u_t|^2\,dx\leq 2\int_0^1(\rho|u|^2|u_x|^2+\rho|\Phi_x|^2+\rho|f|^2$$
$$+\rho^{-1}|[((u_x)^2+\mu_0)^{(p-2)/2}u_x]_x+P_x|^2)dx \tag{24}$$

Then

$$\limsup_{\tau\to 0}\int_0^1 \rho |u_t|^2 dx \le C \tag{25}$$

Taking limit on τ for (23) as τ→0, we get

$$|\sqrt{\rho}u_t(t)|_{L^2}^2 + \int_0^t |u_{xt}|_{L^2}^2(s)ds \le C(1+\int_0^t \Psi^{2\gamma+6}(s)ds) \tag{26}$$

By virtue of (11),(8),(18) and (26), we deduce that

$$|u_x(t)|_{L^p} + |u_{xx}(t)|_{L^2} + |\rho(t)|_{H^1} + |\sqrt{\rho}u_t(t)|_{L^2}$$
$$+\int_0^t |u_{xt}(s)|_{L^2}^2 ds \le C_1 \exp(C_2\int_0^t \Psi^{2\gamma+6}(s)ds) \tag{27}$$

By the definition of $\Psi(t)$, we have

$$\Psi(T) \le C_1 \exp(C_2\int_0^t \Psi^{2\gamma+6}(s)ds) \tag{28}$$

For the inequality (28), if $\int_0^T \Psi^{2\gamma+6}(s)ds < 1$, then we take $T = T_1$; On the other hand, if $\int_0^T \Psi^{2\gamma+6}(s)ds \ge 1$, we can find $t_0 \in (0,T)$, such that $\int_0^T \Psi^{2\gamma+6}(s)ds = 1$.

Choose $T_1 = C_1^{-(2\gamma+6)} e^{-(2\gamma+6)C_2}$, we deduce that $\sup_{0\le t\le T_1} \Psi(t) \le C_1 e^{C_2}$, where C_1, C_2 is positive constant. Then, we obtain the following estimate

$$ess\sup_{0\le t\le T_1}(|\rho(t)|_{H^1} + |u(t)|_{W_0^{1,p}\cap H^2}) + |\sqrt{\rho}u_t(t)|_{L^2} + |\rho_t(t)|_{L^2}$$
$$+\int_0^{T_1} |u_{xt}(s)|_{L^2}^2\, ds \le C \tag{29}$$

Where C is a positive constant, depending only on M_0.

4. Proof of the Existence

In this section, we will use the uniform estimates (33) to prove the existence of the main theorem. Our method that constructed approximate systems is similar to that in [11], we take a semidiscrete Galerkin scheme. We take our basic function space as $X = H_0^1(0,1) \cap H^2(0,1)$ and the finite-dimensional subspaces as

$$X^m = span\{\varphi^1, \varphi^2, \cdots, \varphi^m\} \subset X \cap C^2([0,1]).$$

Here φ^m is the mth eigenfunction of the strongly elliptic operator defined on X. Let ρ_0, u_0 satisfy the hypotheses of Theorem 2.1.1. Asume for the moment that $\rho_0^\delta \in C^1([0,1])$ and $\rho_0^\delta \ge \delta$ in (0, 1) (for some constant $\delta > 0$). We may construct an approximate solution for any $\phi \in X^m, \varphi \in C^2([0,1])$

$$\begin{cases} \int_0^1 (\rho^m u_t^m + \rho^m u^m u_x^m + \rho^m \Phi_x^m - [(u_x^m)^2 + \mu_0)^{(p-2)/2} \\ \quad + u_x^m]_x + P_x^m)\phi dx = \int_0^1 \rho^m f^\delta \phi dx \\ \int_0^1 \rho_t^m \varphi dx + \int_0^1 (\rho^m u^m)_x \varphi dx = 0 \\ \int_0^1 \Phi_{xx}^m \varphi dx = 4\pi g \int_0^1 (\rho^m - \frac{m_0}{|\Omega|})\varphi dx \end{cases}$$

where $f^\delta \in C^1((0,T)\times(0,1))$ and

$$f^\delta \to f \text{ in } L^2(0,T; L^{\frac{2\gamma}{2\gamma-1}}(0,1)).$$

The initial and boundary conditions are

$$u_0^m \equiv \sum_{k=1}^m (u_0, \varphi^k)_{L^2[0,1]} \varphi^k \text{ and } \rho^m(0) = \rho_0^\delta > \delta$$

$$\rho^\delta(0) < |\rho_0|_{L^\infty} + 1,\ |\rho_0^\delta - \rho_0|_{H^1(0,1)} \to 0,$$

$$u^m(0,x) = u^m(1,x) = 0,\ \Phi^m(0) = \Phi^m(1) = 0$$

Under the hypotheses of Theorem 2.1.1, similarly, for any fixed $\delta > 0$, we may get the similar estimate

$$ess\sup_{0\le t\le T_1}(|\rho_\delta^m|_{H^1} + |u_\delta^m|_{W_0^{1,p}\cap H^2}) + |\sqrt{\rho_\delta^m}u_{t\delta}^m|_{L^2} + |\rho_{t\delta}^m|_{L^2}$$
$$+\int_0^{T_1} |u_{xt\delta}^m(s)|_{L^2}^2\, ds \le C \tag{30}$$

Combining the course of estimates and the initial condition of approximate system, we can easily deduce that C is dependent on T, ρ_0, u_0. Moreover, because the constants C are independent of the lower bound of ρ_0. Here C(T) does not depend on δ and m (for any m⩾M), M is dependent on the approximate velocity of initial condition). Thus, we can deduce from the two above estimates that (ρ^m, u^m, Φ^m) conveges, up to an extraction of subsequences, to some limit (ρ_δ, u_δ, Φ_δ) in the obvious weak sense, and there are estimates: δ>0, we may get the similar estimate

$$ess\sup_{0\le t\le T_1}(|\rho_\delta|_{H^1} + |u_\delta|_{W_0^{1,p}\cap H^2}) + |\sqrt{\rho_\delta}u_{t\delta}|_{L^2} + |\rho_{t\delta}|_{L^2}$$
$$+\int_0^{T_1} |u_{xt\delta}|_{L^2}^2\, ds \le C \tag{31}$$

Because C(T) is independent of δ, when δ→0, we can deduct that (ρ_δ, u_δ, Φ_δ) converges, up to an extraction of subsequences, to some limit (ρ, u, Φ) in weak sense and

$$ess\sup_{0\le t\le T_1}(|\rho|_{H^1} + |u|_{W_0^{1,p}\cap H^2}) + |\sqrt{\rho}u_t|_{L^2} + |\rho_t|_{L^2}$$
$$+\int_0^{T_1} |u_{xt}|_{L^2}^2\, ds \le C \tag{32}$$

From the L^p-strong estimates of the equation $(1)_3$, we can easily get the regularity in Theorem 2.1.1.

5. Proof of the Uniqueness

Let (ρ,u,Φ) , $(\bar\rho,\bar u,\bar\Phi)$ be two solutions of the problem (1)-(2). After substituting into the equation respectively, choosing test function $\phi = u-\bar u$, we obtain

$$\frac{1}{2}\int_0^1 \rho(u-\bar u)^2 dx + \int_0^t\int_0^1 ((u_x^2+\mu_0)^{(p-2)/2}u_x$$

$$-(\bar u_x^2+\mu_0)^{(p-2)/2}\bar u_x)(u-\bar u)_x dxds$$

$$\le \int_0^t\int_0^1 \{|\rho-\bar\rho||f-\bar u_t-\bar u\bar u_x-\bar\Phi_x||u-\bar u|$$

$$+|P-\bar P||(u-\bar u)_x|+\rho|(\Phi-\bar\Phi)_x||u-\bar u|$$

$$+\rho|u-\bar u|^2|\bar u_x|\}dxds$$

$$\le \int_0^t \{|\rho-\bar\rho|_{L^2}|f-\bar u_t-\bar u\bar u_x-\bar\Phi_x|_{L^2}|u-\bar u|_{L^\infty}$$

$$+|P-\bar P|_{L^2}|(u-\bar u)_x|_{L^2}+|\rho|_{L^\infty}|(\Phi-\bar\Phi)_x|_{L^2}|u-\bar u|_{L^2}$$

$$+|\sqrt{\rho}(u-\bar u)|_{L^2}^2|\bar u_x|_{L^\infty}\}ds$$

$$\le \int_0^t \{|\rho-\bar\rho|_{L^2}^2|C+C|\bar u_t|_{L^2}^2+C|P-\bar P|_{L^2}^2$$

$$+|\sqrt{\rho}(u-\bar u)|_{L^2}^2+\varepsilon|(u-\bar u)_x|_{L^2}^2\}ds \quad (33)$$

We denote

$$\omega(s)=(s^2+\mu_0)^{\frac{p-2}{2}}s$$

Since

$$\int_0^1((u_x^2+\mu_0)^{(p-2)/2}u_x)-(\bar u_x^2+\mu_0)^{(p-2)/2}\bar u_x)(u-\bar u)_x dx$$

$$=\int_0^1[\int_0^1\omega'(\theta u_x+(1-\theta)\bar u_x)d\theta](u_x-\bar u_x)^2dx$$

$$\omega'(s)=((s^2+\mu_0)^{(p-2)/2}s)'$$

$$=(s_x^2+\mu_0)^{(p-4)/2}((p-1)s^2+\mu_0)$$

$$\ge (s^2+\mu_0)^{(p-2)/2}\ge \mu_0^{(p-2)/2}$$

Consequently (5.1) can be rewritten as

$$|\sqrt{\rho}(u-\bar u)(t)|_{L^2}^2+C\int_0^t|u_x-\bar u_x|_{L^2}^2ds$$

$$\le \int_0^t\{|\rho-\bar\rho|_{L^2}^2\ (C+C|\bar u_t|_{L^2}^2+C|P-\bar P|_{L^2}^2$$

$$+|\sqrt{\rho}(u-\bar u)|_{L^2}^2\}ds \quad (34)$$

Similarly, choosing test function $\varphi=\rho-\bar\rho$, we get

$$\frac{1}{2}\int_0^1(\rho-\bar\rho)^2dx=-\int_0^t\int_0^1(\rho u-\bar\rho\bar u)_x(\rho-\bar\rho)dxds$$

$$=-\int_0^t\int_0^1(\rho_x(u-\bar u)(\rho-\bar\rho)+\rho(u-\bar u)_x(\rho-\bar\rho)$$

$$+\frac{1}{2}\bar u_x(\rho-\bar\rho)^2)dxds$$

$$\le \int_0^t(2|\rho|_{H^1}|u-\bar u|_{L^\infty}|\rho-\bar\rho|_{L^2}+\frac{1}{2}|\bar u_x|_{L^\infty}|\rho-\bar\rho|_{L^2}^2)ds \quad (35)$$

Moreover, from $(1)_1$ we have

$$P_t=-P_xu-\gamma Pu_x,\quad \bar P_t=-\bar P_x\bar u-\gamma\bar P\bar u_x$$

Similarly,

$$(P-\bar P)_t+(P-\bar P)_x\bar u+\bar P_x(u-\bar u)+\gamma(P-\bar P)u_x$$

$$+\gamma\bar P(u-\bar u)_x=0$$

Multiplying it by $(P-\bar P)$ and integrating over Ω_T, we get

$$\frac{1}{2}\int_0^1(P-\bar P)^2dx$$

$$=-\int_0^t\int_0^1(\gamma-\frac{1}{2})(P-\bar P)^2u_x+\bar P_x(u-\bar u)(P-\bar P)$$

$$+\bar P(u-\bar u)_x(P-\bar P)dxds$$

$$\le \int_0^t(C|u_x|_{L^\infty}|P-\bar P|_{L^2}^2+C|\bar P|_{H^1}|u-\bar u|_{L^\infty}|P-\bar P|_{L^2})ds$$

$$\le \int_0^t(C(|u_x|_{L^\infty}+|\bar P|_{H^1}+1)|P-\bar P|_{L^2}^2)ds+\varepsilon|(u-\bar u)_x|_{L^2}^2 \quad (36)$$

From (34)-(36), we obtain

$$|\sqrt{\rho}(u-\bar u)(t)|_{L^2}^2+|(\rho-\bar\rho)(t)|_{L^2}^2+|(P-\bar P)(t)|_{L^2}^2$$

$$+C\int_0^t|(u-\bar u)_x(s)|_{L^2}^2dx$$

$$\le \int_0^t C(1+|\bar u_t|_{L^2}^2+|\bar u_x|_{L^\infty}^2+|\rho|_{H^1}^2+|u_x|_{L^\infty}+|\bar P_x|_{L^2}^2)$$

$$(|\sqrt{\rho}(u-\bar u)(s)|_{L^2}^2+|(\rho-\bar\rho)(s)|_{L^2}^2+|(P-\bar P)(s)|_{L^2}^2)ds \quad (37)$$

And then, Grownwall's inequality yields

$$u = \bar{u},\ \rho = \bar{\rho},$$

From the classical theorems of equation $(1.1)_3$, we get

$$|\Phi - \bar{\Phi}|_{W^{2,2}} = 0.$$

This completes the proof of uniqueness.

6. Conclusion

This paper study the Navier-Stoke-Poisson equations for compressible non-Newtonian fluids in one dimensional bounded intervals. The motion of the fluid is driven by the compressible viscous isentropic flow under the self-gravitational and an external force. The local existence and uniqueness of strong solutions was proved based on some compatibility condition. Through the research of this paper can be for further study of the mechanism of this kind of models and will provide a theoretical basis for further practical applications.

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